

(10.43) $\ln g^*(z) = E_{\pi, \mu, \Lambda} [\ln p(x, z, \pi, \mu, \Lambda)] + c$

ここで

(10.41) $p(x, z, \pi, \mu, \Lambda) = p(x|z, \mu, \Lambda) p(z|\pi) p(\pi) p(\mu|\Lambda) p(\Lambda)$

を代入して

$$\begin{aligned} \ln g^*(z) &= E_{\pi, \mu, \Lambda} [\ln p(x|z, \mu, \Lambda) p(z|\pi) p(\pi) p(\mu|\Lambda) p(\Lambda)] \\ &= E_{\pi, \mu, \Lambda} [\ln p(x|z, \mu, \Lambda)] + E_{\pi, \mu, \Lambda} [\ln p(z|\pi)] + E_{\pi, \mu, \Lambda} [\ln p(\pi) p(\mu|\Lambda) p(\Lambda)] \\ &= E_{\mu, \Lambda} [\ln p(x|z, \mu, \Lambda)] + E_{\pi} [\ln p(z|\pi)] + c \end{aligned}$$

(10.44)

を得る
ここで

(10.39)

$$\begin{aligned} E_{\pi} [\ln p(z|\pi)] &= E_{\pi} \left[\ln \prod_{n=1}^N \prod_{k=1}^K \pi_k^{z_{nk}} \right] \\ &= E_{\pi} \left[\sum_{n=1}^N \sum_{k=1}^K z_{nk} \ln \pi_k \right] \\ &= \sum_{n=1}^N \sum_{k=1}^K z_{nk} E_{\pi_k} [\ln \pi_k] \end{aligned}$$

$$\begin{aligned} E_{\pi} \left[\sum_{n=1}^N \sum_{k=1}^K z_{nk} \ln \pi_k \right] &= \int p(\pi) \sum_{n=1}^N \sum_{k=1}^K z_{nk} \ln \pi_k d\pi \\ &= \int p(\pi) \{ z_{11} \ln \pi_1 + z_{12} \ln \pi_2 + \dots + z_{N1} \ln \pi_1 + z_{N2} \ln \pi_2 + \dots \} d\pi \\ &= z_{11} \int p(\pi) \ln \pi_1 d\pi + \dots \\ &= z_{11} \int p(\pi_1, \pi_2) \ln \pi_1 d\pi_1 d\pi_2 + \dots \\ &= z_{11} \int \ln \pi_1 (\int p(\pi_2, \pi_3) d\pi_2 d\pi_3) d\pi_1 + \dots \\ &= z_{11} \int \ln \pi_1 p(\pi_1) d\pi_1 + \dots \\ &= z_{11} E_{\pi_1} [\ln \pi_1] + \dots \\ &= \sum_n \sum_k z_{nk} E_{\pi_k} [\ln \pi_k] \end{aligned}$$

また

$$\begin{aligned} E_{\mu, \Lambda} [\ln p(x|z, \mu, \Lambda)] &= E_{\mu, \Lambda} \left[\ln \prod_{n=1}^N \prod_{k=1}^K N(z_n | \mu_k, \Lambda_k^{-1})^{z_{nk}} \right] \\ &= E_{\mu, \Lambda} \left[\sum_n \sum_k z_{nk} \ln N(z_n | \mu_k, \Lambda_k^{-1}) \right] \\ &= \sum_n \sum_k z_{nk} E_{\mu_k, \Lambda_k} [\ln N(z_n | \mu_k, \Lambda_k^{-1})] \\ &= \sum_n \sum_k z_{nk} E_{\mu_k, \Lambda_k} \left[\ln \frac{1}{(2\pi)^{d_k}} \frac{1}{|\Lambda_k^{-1}|^{d_k/2}} \exp \left\{ -\frac{1}{2} (z_n - \mu_k)^T \Lambda_k (z_n - \mu_k) \right\} \right] \\ &= \sum_n \sum_k z_{nk} E_{\mu_k, \Lambda_k} \left[-\frac{d_k}{2} \ln(2\pi) + \frac{1}{2} \ln |\Lambda_k| - \frac{1}{2} (z_n - \mu_k)^T \Lambda_k (z_n - \mu_k) \right] \\ &= \sum_n \sum_k z_{nk} \left\{ -\frac{d_k}{2} \ln(2\pi) + \frac{1}{2} E_{\Lambda_k} [\ln |\Lambda_k|] - \frac{1}{2} E_{\mu_k, \Lambda_k} [(z_n - \mu_k)^T \Lambda_k (z_n - \mu_k)] \right\} \end{aligned}$$

$$\begin{aligned} E_{\mu, \Lambda} [f(\mu, \Lambda)] &= \int p(\mu, \Lambda) f(\mu, \Lambda) \prod_k d\mu_k d\Lambda_k \\ &= \int f(\mu, \Lambda) \left(\prod_k p(\mu_k, \Lambda_k) \right) \prod_k d\mu_k d\Lambda_k \\ &= \int f(\mu, \Lambda) p(\mu, \Lambda) d\mu d\Lambda \\ &= E_{p(\mu, \Lambda)} [f(\mu, \Lambda)] \end{aligned}$$

これを代入して

$$\begin{aligned} \ln g^*(z) &= \sum_n \sum_k z_{nk} E_{\pi_k} [\ln \pi_k] + \sum_n \sum_k z_{nk} \left\{ -\frac{d_k}{2} \ln(2\pi) + \frac{1}{2} E_{\Lambda_k} [\ln |\Lambda_k|] - \frac{1}{2} E_{\mu_k, \Lambda_k} [(z_n - \mu_k)^T \Lambda_k (z_n - \mu_k)] \right\} + c \\ &= \sum_n \sum_k z_{nk} \left\{ E_{\pi_k} [\ln \pi_k] - \frac{d_k}{2} \ln(2\pi) + \frac{1}{2} E_{\Lambda_k} [\ln |\Lambda_k|] - \frac{1}{2} E_{\mu_k, \Lambda_k} [(z_n - \mu_k)^T \Lambda_k (z_n - \mu_k)] \right\} + c \\ &= \sum_n \sum_k z_{nk} \ln p_{nk} + c \quad \dots (10.45) \end{aligned}$$

$$\ln p_{nk} = E_{\pi_k} [\ln \pi_k] - \frac{d_k}{2} \ln(2\pi) + \frac{1}{2} E_{\Lambda_k} [\ln |\Lambda_k|] - \frac{1}{2} E_{\mu_k, \Lambda_k} [(z_n - \mu_k)^T \Lambda_k (z_n - \mu_k)] \quad \dots (10.46)$$

を得る

(10.45) の指数をとり

$$\begin{aligned} g^*(z) &= \exp \left(\sum_n \sum_k z_{nk} \ln p_{nk} \right) \exp(c) \\ &= \exp(c) \prod_n \prod_k \exp(\ln p_{nk}^{z_{nk}}) \\ &= \exp(c) \prod_n \prod_k p_{nk}^{z_{nk}} \\ &\propto \prod_n \prod_k p_{nk}^{z_{nk}} \quad \dots (10.47) \end{aligned}$$

を得る

$g^*(z)$ の正規化定数 $A \geq 0$ と

$$\begin{aligned}
 A &= \sum_z \prod_n \prod_k \rho_{nk}^{z_{nk}} \\
 &= \left(\sum_{z_1, k} \rho_{1k}^{z_{1k}} \right) \left(\sum_{z_2, k} \rho_{2k}^{z_{2k}} \right) \dots \quad \leftarrow \sum_{(z_1, z_2)} f(z_1)g(z_2) = \sum_{z_1} \{ f(z_1)g(z_1) + f(z_1)g(z_2) \} = \{ f(z_1)g(z_1) + f(z_1)g(z_2) \} \sum_{z_2} g(z_2) = \sum_{z_1} f(z_1) \sum_{z_2} g(z_2) \\
 &= \left(\sum_k \rho_{1k} \right) \left(\sum_k \rho_{2k} \right) \dots \quad \leftarrow \sum_{z_1} \rho_{1k}^{z_{1k}} = \sum_{z_1} \rho_{1k}^{z_{1k}} \rho_{1k}^{z_{1k}} \dots = \rho_{1k}^1 \rho_{1k}^0 + \rho_{1k}^0 \rho_{1k}^1 + \dots = \rho_{1k} + \rho_{1k} + \dots = \sum_k \rho_{1k} \\
 &= \prod_n \left(\sum_k \rho_{nk} \right)
 \end{aligned}$$

したがって

$$\begin{aligned}
 g^*(z) &= \frac{\prod_n \prod_k \rho_{nk}^{z_{nk}}}{\prod_j \left(\sum_k \rho_{jk} \right)} \\
 &= \frac{\left(\prod_k \rho_{1k}^{z_{1k}} \right) \left(\prod_k \rho_{2k}^{z_{2k}} \right) \dots}{\left(\sum_k \rho_{1k} \right) \left(\sum_k \rho_{2k} \right) \dots} \\
 &= \prod_n \frac{\prod_k \rho_{nk}^{z_{nk}}}{\sum_k \rho_{nk}} \\
 &= \prod_n \prod_k \left(\frac{\rho_{nk}}{\sum_k \rho_{nk}} \right)^{z_{nk}} \quad \leftarrow \prod \left(\frac{\rho_{nk}}{\sum_k \rho_{nk}} \right)^{z_{nk}} = \frac{\rho_{nk}^{z_{nk}} \rho_{nk}^{z_{nk}} \dots}{\left(\sum_k \rho_{nk} \right)^{z_{nk}} \left(\sum_k \rho_{nk} \right)^{z_{nk}} \dots} = \frac{\rho_{nk}^{z_{nk}} \rho_{nk}^{z_{nk}} \dots}{\left(\sum_k \rho_{nk} \right)^{z_{nk} + z_{nk} + \dots}} = \frac{\rho_{nk}^{z_{nk}}}{\sum_k \rho_{nk}} \\
 &= \prod_n \prod_k r_{nk}^{z_{nk}} \quad \dots (10.48)
 \end{aligned}$$

$$r_{nk} = \frac{\rho_{nk}}{\sum_k \rho_{nk}} \quad \dots (10.49)$$

を得る