

$$(10.54) \ln g^*(\pi, \mu, \Lambda) = \ln p(\pi) + \sum_{k=1}^K \ln p(\mu_k, \Lambda_k) + E \left[\ln p(z|w) \right] + \sum_{k=1}^K \sum_{n=1}^N E[z_{nk}] \ln N(z_n | \mu_k, \Lambda_k^{-1}) + C$$

$$(10.59) g^*(\mu_k, \Lambda_k) = N(\mu_k | m_k, (\beta_k \Lambda_k)^{-1}) W(\Lambda_k | w_k, v_k)$$

$$(10.60) \beta_k = \beta_0 + N_k$$

$$(10.61) m_k = \frac{1}{\beta_k} (\beta_0 m_0 + N_k \bar{x}_k)$$

$$(10.62) W_k^{-1} = w_k^{-1} + N_k S_k + \frac{\beta_0 N_k}{\beta_0 + N_k} (\bar{x}_k - m_0)(\bar{x}_k - m_0)^T$$

$$(10.63) v_k = v_0 + N_k$$

$$W(\Lambda | w, v) = B|\Lambda|^{\frac{(v-D-1)/2}{2}} \exp\left(-\frac{1}{2} \text{Tr}(w^T \Lambda)\right) \quad (10.65)$$

$$B(w, v) = (w)^{-\frac{v}{2}} \left(\frac{v}{2} \pi^{\frac{D+1}{2}} \right)^{\frac{D}{2}} \Gamma\left(\frac{v+1-i}{2}\right)^{-1} \quad (10.66)$$

$$(10.54) \text{ a } \mu_k \text{ と } \Lambda_k \text{ だけに依存する項を取り出す} \quad \ln g^*(\mu, \Lambda) = \ln g^*(\mu_1, \Lambda_1) g^*(\mu_2, \Lambda_2) \cdots = \sum_{k=1}^K \ln g^*_k(\mu_k, \Lambda_k)$$

$$\ln g^*(\mu_k, \Lambda_k) = \ln p(\mu_k, \Lambda_k) + \sum_{n=1}^N E[z_{nk}] \ln N(z_n | \mu_k, \Lambda_k^{-1}) + C \leftarrow \text{初期定数}$$

$$= \ln N(m_k | m_0, (\beta_0 \Lambda_k)^{-1}) W(\Lambda_k | w_k, v_k) + \sum_n r_{nk} \ln N(x_n | \mu_k, \Lambda_k^{-1}) + C \quad \xrightarrow{(10.59)}$$

$$= \frac{1}{2} \ln |\Lambda_k| - \frac{1}{2} (\mu_k - m_0)^T \beta_0 \Lambda_k (\mu_k - m_0) + \frac{v_0 - D - 1}{2} \ln |\Lambda_k| - \frac{1}{2} \text{Tr}(w_k^T \Lambda_k) \leftarrow \begin{array}{l} \text{初期定数} \\ (10.65) \text{ と } (10.66) \end{array}$$

$$+ \sum_n r_{nk} \left\{ \frac{1}{2} \ln |\Lambda_k| - \frac{1}{2} (x_n - \mu_k)^T \Lambda_k (x_n - \mu_k) \right\} + C \leftarrow \text{初期定数と } \mu_k, \Lambda_k \text{ が独立な項}$$

① ② $\Sigma \mu_k = \gamma$ にて平方完成

$$- \frac{1}{2} (\mu_k - m_0)^T \beta_0 \Lambda_k (\mu_k - m_0) - \frac{1}{2} \sum_n r_{nk} (x_n - \mu_k)^T \Lambda_k (x_n - \mu_k)$$

$$= -\frac{1}{2} \left\{ \mu_k^T \beta_0 \Lambda_k \mu_k - 2 \mu_k^T \beta_0 \Lambda_k m_0 + m_0^T \beta_0 \Lambda_k m_0 + \sum_n r_{nk} (x_n^T \Lambda_k x_n - 2 \mu_k^T \Lambda_k x_n + \mu_k^T \Lambda_k \mu_k) \right\}$$

$$= -\frac{1}{2} \left\{ \mu_k^T \Lambda_k (\beta_0 + \sum_n r_{nk}) \mu_k - 2 \mu_k^T \Lambda_k (\beta_0 m_0 + \sum_n r_{nk} x_n) + m_0^T \beta_0 \Lambda_k m_0 + \sum_n r_{nk} x_n^T \Lambda_k x_n \right\}$$

$$= -\frac{1}{2} \left\{ \mu_k^T \Lambda_k (\beta_0 + N_k) \mu_k - 2 \mu_k^T \Lambda_k (\beta_0 m_0 + N_k \bar{x}_k) + m_0^T \beta_0 \Lambda_k m_0 + \sum_n r_{nk} x_n^T \Lambda_k x_n \right\} \quad \xrightarrow{(10.51)}$$

$$= -\frac{1}{2} \left\{ \mu_k^T \Lambda_k (\beta_0 + N_k) \mu_k - 2 \mu_k^T \Lambda_k (\beta_0 + N_k) \frac{\beta_0 m_0 + N_k \bar{x}_k}{\beta_0 + N_k} + \left(\frac{\beta_0 m_0 + N_k \bar{x}_k}{\beta_0 + N_k} \right)^T \Lambda_k (\beta_0 + N_k) \left(\frac{\beta_0 m_0 + N_k \bar{x}_k}{\beta_0 + N_k} \right) \right. \\ \left. - \left(\frac{\beta_0 m_0 + N_k \bar{x}_k}{\beta_0 + N_k} \right)^T \Lambda_k (\beta_0 + N_k) \left(\frac{\beta_0 m_0 + N_k \bar{x}_k}{\beta_0 + N_k} \right) + m_0^T \beta_0 \Lambda_k m_0 + \sum_n r_{nk} x_n^T \Lambda_k x_n \right\}$$

$$= -\frac{1}{2} \left\{ \left(\mu_k - \frac{\beta_0 m_0 + N_k \bar{x}_k}{\beta_0 + N_k} \right)^T \Lambda_k (\beta_0 + N_k) \left(\mu_k - \frac{\beta_0 m_0 + N_k \bar{x}_k}{\beta_0 + N_k} \right) \right.$$

$$\left. - \left(\frac{\beta_0 m_0 + N_k \bar{x}_k}{\beta_0 + N_k} \right)^T \Lambda_k (\beta_0 + N_k) \left(\frac{\beta_0 m_0 + N_k \bar{x}_k}{\beta_0 + N_k} \right) + m_0^T \beta_0 \Lambda_k m_0 + \sum_n r_{nk} x_n^T \Lambda_k x_n \right\}$$

$$= -\frac{1}{2} \left\{ (\mu_k - m_k)^T \beta_k \Lambda_k (\mu_k - m_k) - m_k^T \beta_k \Lambda_k m_k + m_0^T \beta_0 \Lambda_k m_0 + \sum_n r_{nk} x_n^T \Lambda_k x_n \right\}$$

$T = T^T = L$

$$\beta_k = \beta_0 + N_k \quad \cdots (10.60)$$

$$m_k = \frac{1}{\beta_k} (\beta_0 m_0 + N_k \bar{x}_k) \quad \cdots (10.61)$$

$\Sigma \ln F'$

$$\begin{aligned}
\ln g^*(\mu_k, \lambda_k) &= \frac{1}{2} \ln |\lambda_k| - \frac{1}{2} (\mu_k - m_k)^T \beta_k \lambda_k (\mu_k - m_k) \\
&\quad + \frac{\nu_k - D - 1}{2} \ln |\lambda_k| + N_k \underbrace{\frac{1}{2} \ln |\lambda_k| - \frac{1}{2} \text{Tr}(W_0^{-1} \lambda_k) + \frac{1}{2} m_k^T \beta_k \lambda_k m_k - \frac{1}{2} m_0^T \beta_0 \lambda_k m_0}_{(10.51)} - \frac{1}{2} \sum_n r_{nk} z_n^T \lambda_k z_n + C \\
&= \frac{1}{2} \ln |\lambda_k| - \frac{1}{2} (\mu_k - m_k)^T \beta_k \lambda_k (\mu_k - m_k) \\
&\quad + \frac{\nu_k + N_k - D - 1}{2} \ln |\lambda_k| - \frac{1}{2} \text{Tr}(W_0^{-1} \lambda_k) + \frac{1}{2} \text{Tr}(m_k m_k^T \beta_k \lambda_k) - \frac{1}{2} \text{Tr}(m_0 m_0^T \beta_0 \lambda_k) - \frac{1}{2} \text{Tr}(\sum_n r_{nk} z_n z_n^T) + C \\
&= \frac{1}{2} \ln |\lambda_k| - \frac{1}{2} (\mu_k - m_k)^T \beta_k \lambda_k (\mu_k - m_k) \\
&\quad + \frac{\nu_k + N_k - D - 1}{2} \ln |\lambda_k| - \frac{1}{2} \text{Tr}(\{W_0^{-1} - m_k m_k^T \beta_k + m_0 m_0^T \beta_0 + \sum_n r_{nk} z_n z_n^T\} \lambda_k) + C
\end{aligned}$$

(3)

$\Sigma = T'$

$$\begin{aligned}
N_k S_k &+ \frac{\beta_0 N_k}{\beta_0 + N_k} (\bar{z}_k - m_0) (\bar{z}_k - m_0)^T \\
&= \sum_n r_{nk} (z_n - \bar{z}_k) (z_n - \bar{z}_k)^T + \frac{\beta_0 N_k}{\beta_k} (\bar{z}_k - m_0) (\bar{z}_k - m_0)^T \\
&= \sum_n r_{nk} z_n z_n^T - (\sum_n r_{nk} z_n)^T \bar{z}_k - \bar{z}_k (\sum_n r_{nk} z_n^T) + (\sum_n r_{nk}) \bar{z}_k \bar{z}_k^T + \frac{\beta_0 N_k}{\beta_k} (\bar{z}_k \bar{z}_k^T - \bar{z}_k m_0^T - m_0 \bar{z}_k^T + m_0 m_0^T) \\
&= \sum_n r_{nk} z_n z_n^T - N_k \bar{z}_k \bar{z}_k^T + \frac{\beta_0 N_k}{\beta_k} (\bar{z}_k \bar{z}_k^T - \bar{z}_k m_0^T - m_0 \bar{z}_k^T + m_0 m_0^T) \\
&= \sum_n r_{nk} z_n z_n^T - \frac{N_k^2}{\beta_k} \sum_n \bar{z}_k \bar{z}_k^T - \frac{\beta_0 N_k}{\beta_k} \bar{z}_k m_0^T - \frac{\beta_0 N_k}{\beta_k} m_0 \bar{z}_k^T + \frac{\beta_0 N_k}{\beta_k} m_0 m_0^T
\end{aligned}$$

$\Sigma = (3) \Sigma (10.2)$

$$\begin{aligned}
&- m_k m_k^T \beta_k + m_0 m_0^T \beta_0 + \sum_n r_{nk} z_n z_n^T \\
&= \sum_n r_{nk} z_n z_n^T - \frac{1}{\beta_k} (\beta_0 m_0 + N_k \bar{z}_k) (\beta_0 m_0 + N_k \bar{z}_k)^T + m_0 m_0^T \beta_0 \\
&= \sum_n r_{nk} z_n z_n^T - \frac{\beta_0^2}{\beta_k} m_0 m_0^T - \frac{\beta_0 N_k}{\beta_k} m_0 \bar{z}_k^T - \frac{\beta_0 N_k}{\beta_k} \bar{z}_k m_0^T - \frac{N_k^2}{\beta_k} \bar{z}_k \bar{z}_k^T + m_0 m_0^T \beta_0 \\
&= \sum_n r_{nk} z_n z_n^T - \frac{N_k^2}{\beta_k} \bar{z}_k \bar{z}_k^T - \frac{\beta_0 N_k}{\beta_k} m_0 \bar{z}_k^T - \frac{\beta_0 N_k}{\beta_k} \bar{z}_k m_0^T + \frac{\beta_0 N_k}{\beta_k} m_0 m_0^T
\end{aligned}$$

$T \neq \Sigma$

(3) $= N_k S_k + \frac{\beta_0 N_k}{\beta_0 + N_k} (\bar{z}_k - m_0) (\bar{z}_k - m_0)^T$

$\Sigma \neq T$

$$\begin{aligned}
\ln g^*(\mu_k, \lambda_k) &= \frac{1}{2} \ln |\lambda_k| - \frac{1}{2} (\mu_k - m_k)^T \beta_k \lambda_k (\mu_k - m_k) \\
&\quad + \frac{\nu_k + N_k - D - 1}{2} \ln |\lambda_k| - \frac{1}{2} \text{Tr}(\{W_k^{-1} + N_k S_k + \frac{\beta_0 N_k}{\beta_0 + N_k} (\bar{z}_k - m_0) (\bar{z}_k - m_0)^T\} \lambda_k) + C \\
&= \frac{1}{2} \ln |\lambda_k| - \frac{1}{2} (\mu_k - m_k)^T \beta_k \lambda_k (\mu_k - m_k) \\
&\quad + \frac{\nu_k - D - 1}{2} \ln |\lambda_k| - \frac{1}{2} \text{Tr}(W_k^{-1} \lambda_k) + C
\end{aligned}$$

$\nu_k = \nu_0 + N_k \quad \dots (10.63)$

$W_k^{-1} = W_0^{-1} + N_k S_k + \frac{\beta_0 N_k}{\beta_0 + N_k} (\bar{z}_k - m_0) (\bar{z}_k - m_0)^T \dots (10.62)$

$\Sigma \neq T$

∴ (n-1)

$$g^*(\mu_k, \Lambda_k) = C \frac{1}{|\Lambda_k|^{\frac{n}{2}}} \exp \left\{ -\frac{1}{2} (\mu_k - m_k)^T \beta_k \Lambda_k (\mu_k - m_k) \right\} |\Lambda_k|^{\frac{n-1}{2}} \exp \left\{ -\frac{1}{2} \text{Tr}(\bar{w}_k^{-1} \Lambda_k) \right\}$$

規格化定数を適当に設定すると

$$g^*(\mu_k, \Lambda_k) = N(\mu_k | M_k, (\beta_k \Lambda_k)^{-1}) W(\Lambda_k | w_k, v_k) \dots (10.59)$$

至得る。