10.16

ノートのタイトル 2016/06/11

$$E_{q(Z)q(\mu,\Lambda)}\left[\ln p(X|Z,\mu,\Lambda)\right] \longleftarrow E_{q(Z,\pi,\mu,\Lambda)}\left[\ln p(X|Z,\mu,\Lambda)\right] = E_{\underset{(10.42)}{q(Z)}\underset{(10.55)}{q(\mu,\Lambda)}}\left[\ln p(X|Z,\mu,\Lambda)\right] = E_{q(Z)q(\mu,\Lambda)}\left[\ln p(X|Z,\mu,\Lambda)\right]\underbrace{E_{q(\pi)}[1]}_{1}$$

$$= E_{q(Z)q(\mu,\Lambda)}\left[\ln\left\{\prod_{n=1}^{N}\prod_{k=1}^{K}\mathcal{N}(x_n|\mu_k,\Lambda_k^{-1})^{z_{nk}}\right\}\right] \quad (10.38)$$

$$= E_{q(Z)q(\mu,\Lambda)}\left[\sum_n\sum_k z_{nk}\ln\mathcal{N}(x_n|\mu_k,\Lambda_k^{-1})\right]$$

$$= \sum_n\sum_k E_{q(Z)q(\mu,\Lambda)}\left[z_{nk}\ln\mathcal{N}(x_n|\mu_k,\Lambda_k^{-1})\right]$$

$$= \sum_n\sum_k E_{q(\mu,\Lambda)}\left[E_{q(Z)}\left[z_{nk}\ln\mathcal{N}(x_n|\mu_k,\Lambda_k^{-1})\right]\right]$$

$$= \sum_n\sum_k E_{q(\mu,\Lambda)}\left[r_{nk}\ln\mathcal{N}(x_n|\mu_k,\Lambda_k^{-1})\right]$$

$$= \sum_n\sum_k r_{nk}E_{q(\mu,\Lambda)}\left[\ln\mathcal{N}(x_n|\mu_k,\Lambda_k^{-1})\right]$$

$$= \sum_n\sum_k r_{nk}E_{q(\mu,\Lambda)}\left[-\frac{D}{2}\ln(2\pi)+\frac{1}{2}\ln|\Lambda_k|-\frac{1}{2}(x_n-\mu_k)^T\Lambda_k(x_n-\mu_k)\right]$$

$$= \sum_n\sum_k r_{nk}\left[-\frac{D}{2}\ln(2\pi)+\frac{1}{2}\ln\tilde{\Lambda}_k - \frac{1}{2}\left\{D\beta_k^{-1}+\nu_k(x_n-m_k)^TW_k(x_n-m_k)\right\}\right]$$

$$= \sum_k \overset{(10.51)}{N_k}\left\{-\frac{D}{2}\ln(2\pi)+\frac{1}{2}\ln\tilde{\Lambda}_k-\frac{1}{2}D\beta_k^{-1}\right\} - \frac{1}{2}\sum_k\nu_k\sum_n r_{nk}(x_n-m_k)^TW_k(x_n-m_k)$$

$$= \sum_k N_k\left\{-\frac{D}{2}\ln(2\pi)+\frac{1}{2}\ln\tilde{\Lambda}_k-\frac{1}{2}D\beta_k^{-1}\right\} - \frac{1}{2}\sum_k\nu_k N_k\left\{\mathrm{Tr}(S_kW_k)+(\bar{x}_k-m_k)^TW_k(\bar{x}_k-m_k)\right\}$$

$$= \frac{1}{2}\sum_k N_k\left[-D\ln(2\pi)+\ln\tilde{\Lambda}_k-D\beta_k^{-1}-\nu_k\left\{\mathrm{Tr}(S_kW_k)+(\bar{x}_k-m_k)^TW_k(\bar{x}_k-m_k)\right\}\right] \quad \cdots (10.71)$$

を得る。

$$E_{q(\mu,\Lambda)}\left[\ln\mathcal{N}(x_n|\mu_k,\Lambda_k^{-1})\right] = \sum_{z_n}q(z_n)z_{nk}\ln\mathcal{N}(x_n|\mu_k,\Lambda_k^{-1})]$$
$$= \sum_{z_1}\cdots\sum_{z_N}q(z_1)\cdots q(z_N)z_{nk}\ln\mathcal{N}(x_n|\mu_k,\Lambda_k^{-1})$$
$$= \sum_{z_1}q(z_1)\cdots\sum_{z_n}q(z_n)\cdots\sum_{z_N}q(z_N)z_{nk}\ln\mathcal{N}(x_n|\mu_k,\Lambda_k^{-1})\cdots$$
$$= \sum_{z_n}\prod_{j=1}^{K}r_{nj}^{z_{nj}}z_{nk}\ln\mathcal{N}(x_n|\mu_k,\Lambda_k^{-1}) \longleftarrow (10.49)\ q(z_n)=\prod_{k=1}^{K}r_{nk}^{z_{nk}}$$
$$= \sum_{z_n}r_{n1}^{z_{n1}}r_{nD}^{z_{nD}}\cdots z_{nk}\ln\mathcal{N}(x_n|\mu_k,\Lambda_k^{-1})$$
$$= r_{nk}\ln\mathcal{N}(x_n|\mu_k,\Lambda_k^{-1})$$

$$\int q(\mu,\Lambda)\ln|\Lambda|d\mu d\Lambda = \int q(\mu|\Lambda)q(\Lambda)\ln|\Lambda|d\mu d\Lambda = \int\left\{\int q(\mu|\Lambda)d\mu\right\}q(\Lambda)\ln|\Lambda|d\Lambda$$
$$= \int q(\Lambda)\ln|\Lambda|d\Lambda = \int\prod_{j=1}^{K}q(\Lambda_j)\ln|\Lambda|d\Lambda = \int q(\Lambda_1)d\Lambda_1\cdots\int q(\Lambda_K)\ln|\Lambda|d\Lambda_K$$
$$= E_{q(\Lambda)}[\ln|\Lambda|] = \ln\tilde{\Lambda}_k \longleftarrow (10.65)$$

$$\int q(\mu,\Lambda)(x_n-\mu_k)^T\Lambda_k(x_n-\mu_k)d\mu d\Lambda = \int\prod_{j=1}^{K}q(\mu_j,\Lambda_j)(x_n-\mu_k)^T\Lambda_k(x_n-\mu_k)d\mu d\Lambda$$
$$= \int q(\mu_k,\Lambda_k)(x_n-\mu_k)^T\Lambda_k(x_n-\mu_k)d\mu_k d\Lambda_k = E_{q(\mu_k,\Lambda_k)}\left[(x_n-\mu_k)^T\Lambda_k(x_n-\mu_k)\right]$$
$$= D\beta_k^{-1}+\nu_k(x_n-m_k)^TW_k(x_n-m_k) \longleftarrow (10.64)$$

$$\mathrm{Tr}(S_kW_k)+(\bar{x}_k-m_k)^TW_k(\bar{x}_k-m_k)$$
$$= \mathrm{Tr}\left\{\frac{1}{N_k}\sum_n r_{nk}(x_n-\bar{x}_k)(x_n-\bar{x}_k)^TW_k\right\}+(\bar{x}_k-m_k)^TW_k(\bar{x}_k-m_k) \quad (10.53)$$
$$= \frac{1}{N_k}\sum_n r_{nk}(x_n-\bar{x}_k)^TW_k(x_n-\bar{x}_k)+(\bar{x}_k-m_k)^TW_k(\bar{x}_k-m_k)$$
$$= \frac{1}{N_k}\sum_n r_{nk}x_n^TW_kx_n-\frac{2}{N_k}\sum_n r_{nk}x_n^TW_k\bar{x}_k+\frac{1}{N_k}\sum_n r_{nk}\bar{x}_k^TW_k\bar{x}_k+\bar{x}_k^TW_k\bar{x}_k-2\bar{x}_k^TW_km_k+m_k^TW_km_k$$
$$= \frac{1}{N_k}\sum_n r_{nk}x_n^TW_kx_n-\frac{2}{N_k}\left(\sum_n r_{nk}x_n^T\right)W_k\bar{x}_k+\frac{1}{N_k}\left(\sum_n r_{nk}\right)\bar{x}_k^TW_k\bar{x}_k+\bar{x}_k^TW_k\bar{x}_k-2\bar{x}_k^TW_km_k+m_k^TW_km_k$$
$$= \frac{1}{N_k}\sum_n r_{nk}x_n^TW_kx_n-2\bar{x}_k^TW_k\bar{x}_k+\bar{x}_k^TW_k\bar{x}_k+\bar{x}_k^TW_k\bar{x}_k-2\bar{x}_k^TW_km_k+m_k^TW_km_k \longleftarrow (10.51),(10.52)$$
$$= \frac{1}{N_k}\sum_n r_{nk}x_n^TW_kx_n-2\bar{x}_k^TW_km_k+m_k^TW_km_k$$
$$= \frac{1}{N_k}\sum_n r_{nk}x_n^TW_kx_n-\frac{2}{N_k}\left(\sum_n r_{nk}x_n^T\right)W_km_k+\frac{1}{N_k}\left(\sum_n r_{nk}\right)m_k^TW_km_k \longleftarrow (10.51),(10.52)$$
$$= \frac{1}{N_k}\sum_n r_{nk}(x_n-m_k)^TW_k(x_n-m_k)$$

Tr の中身

を得る。

また、

$$E_{q(Z)q(\pi)}\left[\ln p(Z|\pi)\right]$$

$$= E_{q(Z)q(\pi)}\left[\ln\prod_{n=1}^{N}\prod_{k=1}^{K}\pi_k^{z_{nk}}\right] \quad (10.37)$$

$$= E_{q(Z)q(\pi)}\left[\sum_n\sum_k z_{nk}\ln\pi_k\right]$$

$$= \sum_n\sum_k E_{q(Z)q(\pi)}\left[z_{nk}\ln\pi_k\right]$$

$$= \sum_n\sum_k E_{q(Z)}[z_{nk}]E_{q(\pi)}[\ln\pi_k]$$

$$= \sum_n\sum_k \overset{(10.50)}{r_{nk}}\ln\overset{(10.66)}{\tilde{\pi}_k} \quad \cdots (10.72)$$

を得る。