

$$(10.70) \quad \mathcal{L} = E[\ln p(x|z, \mu, \Lambda)] + E[\ln p(z|\pi)] + E[\ln p(\pi)] + E[\ln p(\mu, \Lambda)] \\ - E[\ln g(z)] - E[\ln g(\pi)] - E[\ln g(\mu, \Lambda)]$$

潜在変数 z の事後分布を表す

$$p(z, \pi, \mu, \Lambda) = p(z|\pi) p(\pi) p(\mu, \Lambda) \leftarrow (10.41) \text{ 式}$$

$$p(z|\pi) = \prod_k \prod_k \Gamma_k^{z_k} \quad \cdots (10.47)$$

$$p(\pi) = \text{Dir}(\pi|\alpha_0) \quad \cdots (10.39)$$

$$p(\mu, \Lambda) = \prod_k N(\mu_k | m_0, (\beta_0 \Lambda_k)^{-1}) W(\Lambda_k | W_0, V_0) \cdots (10.40)$$

この式は共役事前分布を持つ。事後分布 $p(z, \pi, \mu, \Lambda | X)$ も同じ関係性を持つ

これをふまえて事後分布を導く(似似)

$$g(z, \pi, \mu, \Lambda) = g(z) g(\pi) \prod_k g(\mu_k, \Lambda_k) \cdots (10.42), (10.55)$$

$$g(z) = \prod_n \prod_k \Gamma_{nk}^{z_k} \quad \cdots (10.48)$$

$$g(\pi) = \text{Dir}(\pi|\alpha) \quad \cdots (10.57)$$

$$g(\mu_k, \Lambda_k) = N(\mu_k | m_k, (\beta_k \Lambda_k)^{-1}) W(\Lambda_k | W_k, V_k) \cdots (10.59)$$

左の式と仮定可

要分明解 $(10.70) \leftarrow (10.71) - (10.77)$ を代入して得られる

$$\begin{aligned} \mathcal{L} &= E[\ln p(x|z, \mu, \Lambda)] + E[\ln p(z|\pi)] + E[\ln p(\pi)] + E[\ln p(\mu, \Lambda)] \\ &\quad - E[\ln g(z)] - E[\ln g(\pi)] - E[\ln g(\mu, \Lambda)] \leftarrow (10.70) \\ &= \frac{1}{2} \sum_k N_k \left\{ \ln \tilde{\Lambda}_k - D \beta_k^{-1} - V_k \text{Tr}(S_k W_k) - V_k (\bar{x}_k - m_k)^T W_k (\bar{x}_k - m_k) - D \ln(2\pi) \right\} \leftarrow (10.72) \\ &\quad + \sum_n \sum_k r_{nk} \ln \tilde{\Gamma}_{nk} + \ln C(\alpha_0) + (\alpha_0 - 1) \sum_k \ln \tilde{\Gamma}_{kk} \leftarrow (10.73) \\ &\quad + \frac{1}{2} \sum_k \left\{ D \ln \left(\frac{\beta_0}{2\pi} \right) + \ln \tilde{\Lambda}_k - \frac{D \beta_0}{\beta_k} - \beta_0 V_k (m_k - m_0)^T W_k (m_k - m_0) \right\} \leftarrow (10.74) \\ &\quad + \ln B(W_0, V_0) + \frac{V_0 - D - 1}{2} \sum_k \ln \tilde{\Lambda}_k - \frac{1}{2} \sum_k V_k \text{Tr}(W_0^{-1} W_k) \leftarrow (10.75) \\ &\quad - \sum_n \sum_k r_{nk} \ln r_{nk} - \sum_k (\alpha_k - 1) \ln \tilde{\Gamma}_{kk} - \ln C(\alpha) \leftarrow (10.76) \\ &\quad - \sum_k \left\{ \frac{1}{2} \ln \tilde{\Lambda}_k + \frac{D}{2} \ln \left(\frac{\beta_k}{2\pi} \right) - \frac{D}{2} - H[\Lambda_k] \right\} \leftarrow (10.77) \end{aligned}$$

L E 7 L 整理稿

$$L = \frac{1}{2} \sum_k N_k \left\{ \ln \tilde{\pi}_k - D \beta_k^{-1} - V_k \text{Tr}(S_k W_k) - V_k (\bar{x}_k - m_k)^T W_k (\bar{x}_k - m_k) - D \ln(2\pi) \right\}$$

$$+ \sum_n \sum_k r_{nk} \ln \tilde{\pi}_k + \ln C(\alpha_0) + (\alpha_0 - 1) \sum_k \ln \tilde{\pi}_k$$

$$+ \frac{1}{2} \sum_k \left\{ D \ln \left(\frac{\beta_0}{2\pi} \right) + \ln \tilde{\pi}_k - \frac{D\beta_0}{\beta_k} - \beta_0 V_k (m_k - m_0)^T W_k (m_k - m_0) \right\}$$

$$+ K \ln B(W_0, V_0) + \frac{V_0 - D - 1}{2} \sum_k \ln \tilde{\pi}_k - \frac{1}{2} \sum_k V_k \text{Tr}(W_0^{-1} W_k)$$

$$- \sum_n \sum_k r_{nk} \ln r_{nk} - \sum_k (\alpha_k - 1) \ln \tilde{\pi}_k - \ln C(\alpha)$$

$$- \sum_k \left\{ \frac{1}{2} \ln \tilde{\pi}_k + \frac{D}{2} \ln \left(\frac{\beta_k}{2\pi} \right) - \frac{D}{2} + \ln B(W_k, V_k) + \frac{V_k - D - 1}{2} \ln \tilde{\pi}_k - \frac{V_k D}{2} \right\}$$

$$= \frac{1}{2} \sum_k \ln \tilde{\pi}_k \{ N_k + 1 + (V_0 - D - 1) - 1 - (V_k - D - 1) \}$$

$$+ \sum_k \ln \tilde{\pi}_k \left\{ \sum_n r_{nk} + (\alpha_0 - 1) - (\alpha_k - 1) \right\}$$

$$+ \frac{1}{2} \sum_k \left\{ \beta_k^{-1} (-N_k D - D\beta_0) - D \ln \left(\frac{\beta_k}{2\pi} \right) \right\}$$

$$+ \frac{1}{2} \sum_k N_k \left\{ -V_k \text{Tr}(S_k W_k) - V_k (\bar{x}_k - m_k)^T W_k (\bar{x}_k - m_k) \right\}$$

$$+ \frac{1}{2} \sum_k \left\{ -V_k \text{Tr}(W_0^{-1} W_k) - \beta_0 V_k (m_k - m_0)^T W_k (m_k - m_0) \right\}$$

$$- \sum_k \left\{ \ln B(W_k, V_k) - \frac{V_k D}{2} \right\}$$

$$- \frac{1}{2} \sum_k N_k D \ln(2\pi) + \ln C(\alpha_0) + \frac{1}{2} \sum_k D \ln \left(\frac{\beta_0}{2\pi} \right) + K \ln B(W_0, V_0)$$

$$- \sum_n \sum_k r_{nk} \ln r_{nk} - \ln C(\alpha) - \sum_k \frac{D}{2}$$

$$= \frac{1}{2} \sum_k \ln \tilde{\pi}_k (N_k + V_0 - V_k)$$

$$+ \sum_k \ln \tilde{\pi}_k (N_k + \alpha_0 - \alpha_k)$$

$$- \frac{D}{2} \sum_k \left\{ \beta_k^{-1} (N_k + \beta_0) + \ln \left(\frac{\beta_k}{2\pi} \right) \right\}$$

$$- \frac{1}{2} \sum_k N_k V_k \left\{ \text{Tr}(S_k W_k) + (\bar{x}_k - m_k)^T W_k (\bar{x}_k - m_k) \right\}$$

$$- \frac{1}{2} \sum_k V_k \left\{ \text{Tr}(W_0^{-1} W_k) + \beta_0 (m_k - m_0)^T W_k (m_k - m_0) \right\}$$

$$- \sum_k \left\{ \ln B(W_k, V_k) - \frac{V_k D}{2} \right\}$$

$$- \sum_n \sum_k r_{nk} \ln r_{nk} - \ln C(\alpha)$$

$$- \frac{1}{2} N D \ln(2\pi) + \ln C(\alpha_0) + \frac{1}{2} K D \ln \left(\frac{\beta_0}{2\pi} \right) + K \ln B(W_0, V_0) + \frac{D K}{2}$$

(B, P2), (10.65)

L の停留条件から $1^{\circ} \leq x - t$ の更新式を得る。

$$1^{\circ} \leq x - t \text{ は } r_{ik}, \underline{\alpha_k, \beta_k, m_k, w_k, v_k}$$

EXP.の下限を 1° とする

($\alpha_k = \gamma_{1/2}$) — 目標 (10.58) $\alpha_k = \alpha_0 + N_k$

L の α_k を含む項

$$\begin{aligned} L &= \sum_k \ln \tilde{T}_k (N_k + \alpha_0 - \alpha_k) - \ln C(\alpha) + \text{const} \\ &= \sum_k \{4(\alpha_k) - 4(\hat{\alpha})\} (N_k + \alpha_0 - \alpha_k) - \ln \Gamma(\hat{\alpha}) + \sum_{k=1}^K \ln \Gamma(\alpha_k) + \text{const} \\ &= \sum_k 4(\alpha_k) (N_k + \alpha_0 - \alpha_k) - 4(\hat{\alpha}) \sum_k (N_k + \alpha_0 - \alpha_k) - \ln \Gamma(\hat{\alpha}) + \sum_{k=1}^K \ln \Gamma(\alpha_k) + \text{const} \end{aligned}$$

停留条件 12.

$$0 = \frac{\partial L}{\partial \alpha_k}$$

$$\begin{aligned} 0 &= \frac{\partial 4(\alpha_k)}{\partial \alpha_k} (N_k + \alpha_0 - \alpha_k) - 4(\alpha_k) - \frac{\partial \hat{\alpha}}{\partial \alpha_k} \frac{\partial 4(\hat{\alpha})}{\partial \hat{\alpha}} \sum_k (N_k + \alpha_0 - \alpha_k) + 4(\hat{\alpha}) \\ &\quad - \frac{\partial \hat{\alpha}}{\partial \alpha_k} \frac{\partial}{\partial \hat{\alpha}} \ln \Gamma(\hat{\alpha}) + \frac{\partial}{\partial \alpha_k} \ln \Gamma(\alpha_k) \\ &= \frac{\partial 4(\alpha_k)}{\partial \alpha_k} (N_k + \alpha_0 - \alpha_k) - 4(\alpha_k) - \frac{\partial 4(\hat{\alpha})}{\partial \hat{\alpha}} \sum_k (N_k + \alpha_0 - \alpha_k) + 4(\hat{\alpha}) \\ &\quad - 4(\hat{\alpha}) + 4(\alpha_k) \end{aligned}$$

$$\begin{aligned} \hat{\alpha} &= \frac{1}{K} \sum_{k=1}^K \alpha_k \quad \frac{\partial \hat{\alpha}}{\partial \alpha_k} = 1 \\ \hat{\alpha} &\in (0, 2) \subset \mathbb{R}^+ \end{aligned}$$

$$\begin{aligned} \frac{\partial \hat{\alpha}}{\partial \alpha_k} \frac{\partial}{\partial \hat{\alpha}} \ln \Gamma(\hat{\alpha}) &= 4(\hat{\alpha}) \\ \frac{\partial}{\partial \alpha_k} \ln \Gamma(\hat{\alpha}) &= 4(\alpha_k) \end{aligned}$$

したがって

$$N_k + \alpha_0 - \alpha_k = 0$$

となる成立条件。

つまり

$$\alpha_k = N_k + \alpha_0$$

α_k とき L は停留する。

(β_k は γ_{12}) — 目標 (10.60) $\beta_k = \beta_0 + N_k$

より β_k を含む項は

$$\mathcal{L} = -\frac{D}{2} \left\{ \sum_k \beta_k^{-1} (N_k + \beta_0) + \ln \frac{\beta_k}{2\pi} \right\} + \text{const}$$

停留条件は

$$0 = \frac{\partial \mathcal{L}}{\partial \beta_k}$$

$$= -\frac{D}{2} \left\{ \frac{-1}{\beta_k^2} (N_k + \beta_0) + \frac{1}{\beta_k} \right\}$$

$$\therefore 0 = -(N_k + \beta_0) + \beta_k$$

$$\therefore \beta_k = N_k + \beta_0$$

を得る。

(m_k は γ_{12}) — 目標 (10.61) $m_k = \frac{1}{\beta_k} (\beta_0 m_0 + N_k \bar{x}_k)$, $\beta_k = \beta_0 + N_k$

より m_k を含む項は

$$\mathcal{L} = -\frac{1}{2} \left\{ \sum_k N_k V_k (\bar{x}_k - m_k)^T W_k (\bar{x}_k - m_k) + \sum_k V_k \beta_0 (m_k - m_0)^T W_k (m_k - m_0) \right\} + \text{const}$$

停留条件は

$$0 = \frac{\partial \mathcal{L}}{\partial m_k}$$

$$= -\frac{1}{2} \left\{ N_k V_k (-2W_k \bar{x}_k + 2W_k m_k) + V_k \beta_0 (-2W_k m_0 + 2W_k m_k) \right\} \quad \leftarrow$$

$$= N_k V_k W_k (\bar{x}_k - m_k) + V_k \beta_0 W_k (m_0 - m_k)$$

$$= V_k W_k \{ N_k (\bar{x}_k - m_k) + \beta_0 (m_0 - m_k) \}$$

$$\therefore 0 = N_k \bar{x}_k + \beta_0 m_0 - (N_k + \beta_0) m_k$$

∴

$$m_k = \frac{1}{N_k + \beta_0} (N_k \bar{x}_k + \beta_0 m_0)$$

$$= \frac{1}{\beta_k} (N_k \bar{x}_k + \beta_0 m_0) \quad \leftarrow \beta_k = N_k + \beta_0 \text{ を代入}$$

を得る。

$$(W_k \geq 0) \text{ — 目標 (10.62) } W_k = W_0^{-1} + N_k S_k + \frac{\beta_0 N_k}{\beta_0 + N_k} (\bar{x}_k - m_0)(\bar{x}_k - m_0)^T$$

\checkmark W_k は既存項

$$L = \frac{1}{2} \sum_k \ln \tilde{\lambda}_k (N_k + V_0 - V_k)$$

$$- \frac{1}{2} \sum_k N_k V_k \{ \text{Tr}(S_k W_k) + (\bar{x}_k - m_k)^T W_k (\bar{x}_k - m_k) \}$$

$$- \frac{1}{2} \sum_k V_k \{ \text{Tr}(W_0^{-1} W_k) + \beta_0 (m_k - m_0)^T W_k (m_k - m_0) \}$$

$$- \sum_k \ln B(W_k, V_k) + \text{const}$$

← (10.65)

$$= \frac{1}{2} \sum_k \left\{ \sum_{i=1}^D \left(\frac{V_k + 1 - i}{2} \right) + D \ln 2 + \ln |W_k| \right\} (N_k + V_0 - V_k)$$

$$- \frac{1}{2} \sum_k N_k V_k \{ \text{Tr}(S_k W_k) + (\bar{x}_k - m_k)^T W_k (\bar{x}_k - m_k) \}$$

$$- \frac{1}{2} \sum_k V_k \{ \text{Tr}(W_0^{-1} W_k) + \beta_0 (m_k - m_0)^T W_k (m_k - m_0) \}$$

$$- \sum_k \ln B(W_k, V_k) + \text{const}$$

停留条件

$$\begin{aligned} 0 &= \frac{\partial L}{\partial W_k} \\ &= \frac{1}{2} W_k^{-1} (N_k + V_0 - V_k) \\ &\quad - \frac{1}{2} N_k V_k \{ S_k + (\bar{x}_k - m_k)(\bar{x}_k - m_k)^T \} \\ &\quad - \frac{1}{2} V_k \{ W_0^{-1} + \beta_0 (m_k - m_0)(m_k - m_0)^T \} \\ &\quad + \frac{V_k}{2} W_k^{-1} \end{aligned}$$

$$\begin{aligned} \ln B(W_k, V_k) &= \ln |W_k|^{V_k} \left(\frac{N_k}{2} \pi^{\frac{D(D-1)}{2}} \prod_{i=1}^D P \left(\frac{V_k + 1 - i}{2} \right) \right)^{-1} \\ \frac{\partial}{\partial W_k} \ln B(W_k, V_k) &= -\frac{1}{2} \frac{2}{N_k} \ln |W_k| = -\frac{V_k}{N_k} W_k^{-1} \end{aligned}$$

∴ (10.61)

$$0 = W_k^{-1} (N_k + V_0) - N_k V_k \{ S_k + (\bar{x}_k - m_k)(\bar{x}_k - m_k)^T \} - V_k \{ W_0^{-1} + \beta_0 (m_k - m_0)(m_k - m_0)^T \}$$

$$\begin{aligned} W_k^{-1} &= \frac{V_k}{N_k + V_0} W_0^{-1} + \frac{N_k V_k}{N_k + V_0} S_k + \frac{N_k V_k}{N_k + V_0} (\bar{x}_k - m_k)(\bar{x}_k - m_k)^T + \frac{V_k \beta_0}{N_k + V_0} (m_k - m_0)(m_k - m_0)^T \\ &= \frac{V_k}{N_k + V_0} \{ W_0^{-1} + N_k S_k + N_k (\bar{x}_k - m_k)(\bar{x}_k - m_k)^T + \beta_0 (m_k - m_0)(m_k - m_0)^T \} \end{aligned}$$

∴ 2nd 次の m_k の更新式を便り

$$W_k^{-1} = \frac{V_k}{N_k + V_0} \{ W_0^{-1} + N_k S_k + \frac{N_k \beta_0}{N_k + \beta_0} (\bar{x}_k - m_0)(\bar{x}_k - m_0)^T \}$$

を得る。



$$\begin{aligned} &N_k (\bar{x}_k - m_k)(\bar{x}_k - m_k)^T + \beta_0 (m_k - m_0)(m_k - m_0)^T \\ &= N_k \bar{x}_k \bar{x}_k^T - N_k \bar{x}_k m_k^T - N_k m_k \bar{x}_k^T + N_k m_k m_k^T \\ &\quad + \beta_0 m_k m_k^T - \beta_0 m_k m_k^T - \beta_0 m_k m_k^T + \beta_0 m_k m_k^T \\ &= N_k \bar{x}_k \bar{x}_k^T + \beta_0 m_k m_k^T \\ &\quad - (N_k \bar{x}_k + \beta_0 m_k) m_k^T - m_k (N_k \bar{x}_k + \beta_0 m_k)^T \\ &\quad + (N_k + \beta_0) m_k m_k^T \\ &= N_k \bar{x}_k \bar{x}_k^T + \beta_0 m_k m_k^T - \frac{1}{N_k + \beta_0} (N_k \bar{x}_k + \beta_0 m_k)(N_k \bar{x}_k + \beta_0 m_k)^T \\ &= \frac{1}{N_k + \beta_0} \{ (N_k + \beta_0) N_k \bar{x}_k \bar{x}_k^T + (N_k + \beta_0) \beta_0 m_k m_k^T - (N_k \bar{x}_k + \beta_0 m_k)(N_k \bar{x}_k + \beta_0 m_k)^T \} \\ &= \frac{1}{N_k + \beta_0} \{ (N_k + \beta_0) N_k \bar{x}_k \bar{x}_k^T + (N_k + \beta_0) \beta_0 m_k m_k^T \\ &\quad - (N_k^2 \bar{x}_k \bar{x}_k^T + N_k \beta_0 \bar{x}_k m_k^T + \beta_0 N_k m_k \bar{x}_k^T + \beta_0^2 m_k m_k^T) \} \\ &= \frac{1}{N_k + \beta_0} (N_k \bar{x}_k \bar{x}_k^T + m_k m_k^T - \bar{x}_k \bar{x}_k^T - m_k \bar{x}_k^T) \\ &= \frac{N_k \beta_0}{N_k + \beta_0} (\bar{x}_k \bar{x}_k^T - m_k \bar{x}_k^T) \\ &= \frac{N_k \beta_0}{N_k + \beta_0} (\bar{x}_k - m_0)(\bar{x}_k - m_0)^T \end{aligned}$$

$(V_k = \gamma_{1,2})$ - 目標 (10.63) $V_k = V_0 + N_k$

$\Leftrightarrow V_k \in \text{簇} \Rightarrow \text{直}$

$$\mathcal{L} = \frac{1}{2} \sum_k \ln \tilde{\lambda}_k (N_k + V_0 - V_k)$$

$$- \frac{1}{2} \sum_k N_k V_k \left\{ \text{Tr}(S_k W_k) + (\bar{x}_k - m_k)^T W_k (\bar{x}_k - m_k) \right\}$$

$$- \frac{1}{2} \sum_k V_k \left\{ \text{Tr}(W_0^{-1} W_k) + \beta_0 (m_k - m_0)^T W_k (m_k - m_0) \right\}$$

$$- \sum_k \left\{ \ln B(W_k, V_k) - \frac{V_k D}{2} \right\} + \text{const}$$

$$= \frac{1}{2} \sum_k \left\{ \sum_{i=1}^D \frac{1}{2} \left(\frac{V_k + 1 - i}{2} \right) + D \ln 2 + \ln |W_k| \right\} (N_k + V_0 - V_k)$$

$$- \frac{1}{2} \sum_k N_k V_k \left\{ \text{Tr}(S_k W_k) + (\bar{x}_k - m_k)^T W_k (\bar{x}_k - m_k) \right\}$$

$$- \frac{1}{2} \sum_k V_k \left\{ \text{Tr}(W_0^{-1} W_k) + \beta_0 (m_k - m_0)^T W_k (m_k - m_0) \right\}$$

$$- \sum_k \left\{ \ln |W_k|^{\frac{V_k}{2}} \left(2^{\frac{V_k D}{2}} \pi^{\frac{D(D-1)}{4}} \prod_{i=1}^D \Gamma \left(\frac{V_k + 1 - i}{2} \right) \right)^{-1} - \frac{V_k D}{2} \right\} + \text{const}$$

停留条件は

$$0 = \frac{\partial \mathcal{L}}{\partial V_k}$$

$$= \frac{1}{2} \sum_{i=1}^D \frac{\partial}{\partial V_k} \left[\frac{1}{2} \left(\frac{V_k + 1 - i}{2} \right) (N_k + V_0 - V_k) - \frac{1}{2} \left\{ \sum_{i=1}^D \frac{1}{2} \left(\frac{V_k + 1 - i}{2} \right) + D \ln 2 + \ln |W_k| \right\} \right]$$

$$- \frac{1}{2} N_k \left\{ \text{Tr}(S_k W_k) + (\bar{x}_k - m_k)^T W_k (\bar{x}_k - m_k) \right\}$$

$$- \frac{1}{2} \left\{ \text{Tr}(W_0^{-1} W_k) + \beta_0 (m_k - m_0)^T W_k (m_k - m_0) \right\}$$

$$- \left\{ -\frac{1}{2} \ln |W_k| - \frac{D}{2} \ln 2 - \sum_{i=1}^D \frac{1}{2} \left(\frac{V_k + 1 - i}{2} \right) \right\} + \frac{1}{2} D$$

$$= \frac{1}{2} \sum_{i=1}^D \frac{\partial}{\partial V_k} \left[\frac{1}{2} \left(\frac{V_k + 1 - i}{2} \right) (N_k + V_0 - V_k) \right]$$

$$- \frac{1}{2} N_k \left\{ \text{Tr}(S_k W_k) + (\bar{x}_k - m_k)^T W_k (\bar{x}_k - m_k) \right\}$$

$$- \frac{1}{2} \left\{ \text{Tr}(W_0^{-1} W_k) + \beta_0 (m_k - m_0)^T W_k (m_k - m_0) \right\} + \frac{1}{2} D$$

$$= \frac{1}{2} \left[\sum_{i=1}^D \frac{\partial}{\partial V_k} \left(\frac{1}{2} \left(\frac{V_k + 1 - i}{2} \right) (N_k + V_0 - V_k) \right) \right.$$

$$- \text{Tr}(N_k S_k W_k) - \text{Tr}(N_k (\bar{x}_k - m_k) (\bar{x}_k - m_k)^T W_k)$$

$$- \text{Tr}(W_0^{-1} W_k) - \text{Tr}(\beta_0 (m_k - m_0) (m_k - m_0)^T W_k) + D \Big]$$

5, 2 条件式は

$$0 = \sum_{i=1}^D \frac{\partial}{\partial V_k} \left[\frac{1}{2} \left(\frac{V_k + 1 - i}{2} \right) (N_k + V_0 - V_k) \right]$$

$$- \text{Tr} \left(\underbrace{(N_k S_k + N_k (\bar{x}_k - m_k) (\bar{x}_k - m_k)^T + W_0^{-1} + \beta_0 (m_k - m_0) (m_k - m_0)^T) W_k}_{(1)} \right) + D$$

27.3.0

ここで上の m_k, w_k の更新式を使うと

$$\begin{aligned}
 ① &= N_k \bar{s}_k + N_k \bar{\gamma}_k \bar{\gamma}_k^T - N_k \bar{\gamma}_k \bar{m}_k^T - N_k \bar{m}_k \bar{\gamma}_k^T + N_k \bar{m}_k \bar{m}_k^T + \bar{w}_o^{-1} + \beta_o \bar{m}_k \bar{m}_k^T - \beta_o \bar{m}_k \bar{m}_o^T - \beta_o \bar{m}_o \bar{m}_k^T + \beta_o \bar{m}_o \bar{m}_o^T \\
 &= \bar{w}_o^{-1} + N_k \bar{s}_k + N_k \bar{\gamma}_k \bar{\gamma}_k^T - (N_k \bar{\gamma}_k + \beta_o \bar{m}_o) \bar{m}_k^T - m_k (N_k \bar{\gamma}_k + \beta_o \bar{m}_o)^T + (N_k + \beta_o) \bar{m}_k \bar{m}_k^T + \beta_o \bar{m}_o \bar{m}_o^T \\
 &= \bar{w}_o^{-1} + N_k \bar{s}_k + N_k \bar{\gamma}_k \bar{\gamma}_k^T - \frac{1}{N_k + \beta_o} (N_k \bar{\gamma}_k + \beta_o \bar{m}_o) (N_k \bar{\gamma}_k + \beta_o \bar{m}_o)^T + \beta_o \bar{m}_o \bar{m}_o^T \quad m_k = \frac{1}{N_k + \beta_o} (N_k \bar{\gamma}_k + \beta_o \bar{m}_o) \bar{m}_k^T \\
 &= \bar{w}_o^{-1} + N_k \bar{s}_k + \frac{1}{N_k + \beta_o} \left\{ (N_k + \beta_o) N_k \bar{\gamma}_k \bar{\gamma}_k^T - N_k^2 \bar{\gamma}_k \bar{\gamma}_k^T - N_k \beta_o \bar{\gamma}_k \bar{m}_o - \beta_o N_k \bar{m}_o \bar{\gamma}_k^T - \beta_o^2 \bar{m}_o \bar{m}_o^T + (N_k + \beta_o) \beta_o \bar{m}_o \bar{m}_o^T \right\} \\
 &= \bar{w}_o^{-1} + N_k \bar{s}_k + \frac{N_k \beta_o}{N_k + \beta_o} \left(\bar{\gamma}_k \bar{\gamma}_k^T - \bar{\gamma}_k \bar{m}_o^T - \bar{m}_o \bar{\gamma}_k^T + \bar{m}_o \bar{m}_o^T \right) \\
 &= \bar{w}_o^{-1} + N_k \bar{s}_k + \frac{N_k \beta_o}{N_k + \beta_o} (\bar{\gamma}_k - \bar{m}_o)(\bar{\gamma}_k - \bar{m}_o)^T \\
 &= \frac{N_k + \nu_o}{\nu_k} \bar{W}_k^{-1} \quad \leftarrow \bar{w}_o^{-1} = \frac{\nu_k}{N_k + \nu_o} \left\{ \bar{w}_o^{-1} + N_k \bar{s}_k + \frac{N_k \beta_o}{N_k + \beta_o} (\bar{\gamma}_k - \bar{m}_o)(\bar{\gamma}_k - \bar{m}_o)^T \right\}
 \end{aligned}$$

二つの条件式についてまとめて

$$\begin{aligned}
 D &= \sum_{i=1}^D \frac{\partial}{\partial \nu_k} 4 \left(\frac{\nu_k + 1 - i}{2} \right) (N_k + \nu_o - \nu_k) \\
 &\quad - \text{Tr} \left(\frac{N_k + \nu_o}{\nu_k} \bar{W}_k^{-1} W_k \right) + D \\
 &= \sum_{i=1}^D \frac{\partial}{\partial \nu_k} 4 \left(\frac{\nu_k + 1 - i}{2} \right) (N_k + \nu_o - \nu_k) - \frac{N_k + \nu_o}{\nu_k} \underbrace{\text{Tr}(W_k^{-1} W_k)}_D + D
 \end{aligned}$$

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$$D = \sum_{i=1}^D \frac{\partial}{\partial \nu_k} 4 \left(\frac{\nu_k + 1 - i}{2} \right) (N_k + \nu_o - \nu_k) - \left(\frac{N_k + \nu_o}{\nu_k} - 1 \right) D$$

二の条件は

$$\nu_k = N_k + \nu_o$$

を満たす

(1) $r_{nk} \in \gamma \cup \bar{\gamma}$ 一目標 (10.67) $r_{nk} \propto \tilde{\pi}_k \tilde{\lambda}_k^k \exp\left\{-\frac{D}{2} - \frac{W_k}{2}(z_n - m_k)^T W_k (z_n - m_k)\right\}$

r_{nk} の制約条件

$$\sum_k r_{nk} = N$$

左辺の r_{nk} は係数項と制約条件によりランジターナ

$$L = \frac{1}{2} \sum_k \ln \tilde{\lambda}_k N_k + \sum_k \ln \tilde{\pi}_k N_k - \frac{D}{2} \sum_k \beta_k^{-1} N_k$$

$$- \frac{1}{2} \sum_k N_k V_k \{ \text{Tr}(S_k W_k) + (\bar{x}_k - m_k)^T W_k (\bar{x}_k - m_k) \}$$

$$- \sum_n \sum_k r_{nk} \ln r_{nk} - \frac{1}{2} ND \ln(2\pi) + \text{const} + \lambda \left(\sum_n \sum_k r_{nk} - N \right)$$

式23。

停留条件は

$$0 = \frac{\partial L}{\partial r_{nk}}$$

$$(10.61) \text{ 例)} \\ N_k = \frac{1}{r_{nk}} \\ \frac{2N_k}{r_{nk}} = 1$$

$$= \frac{1}{2} \ln \tilde{\lambda}_k + \ln \tilde{\pi}_k - \frac{D}{2} \beta_k^{-1}$$

Trと併せては分子で
分母で同じ式

$$- \frac{1}{2} V_k \{ \text{Tr}(S_k W_k) + (\bar{x}_k - m_k)^T W_k (\bar{x}_k - m_k) \}$$

$$- \frac{1}{2} N_k V_k \left\{ \text{Tr}\left(\frac{\partial}{\partial r_{nk}} S_k W_k\right) + \frac{2}{\partial r_{nk}} (\bar{x}_k - m_k)^T W_k (\bar{x}_k - m_k) \right\}$$

$$= \frac{1}{2} \ln \tilde{\lambda}_k + \ln \tilde{\pi}_k - \frac{D}{2} \beta_k^{-1}$$

$$- \frac{1}{2} V_k \{ \text{Tr}(S_k W_k) + (\bar{x}_k - m_k)^T W_k (\bar{x}_k - m_k) \}$$

$$- \frac{1}{2} N_k V_k \left[\text{Tr}\left[\frac{1}{N_k} \{ S_k + (z_n - \bar{z}_k)(z_n - \bar{z}_k)^T \} W_k\right] \right.$$

$$\left. + \frac{2}{N_k} (\bar{x}_k - m_k)^T W_k (z_n - \bar{x}_k) \right]$$

$$- (\ln r_{nk} + 1) - \frac{D}{2} \ln(2\pi) + \lambda$$

$$= \frac{1}{2} \ln \tilde{\lambda}_k + \ln \tilde{\pi}_k - \frac{D}{2} \beta_k^{-1}$$

$$- \frac{1}{2} V_k \{ \text{Tr}(S_k W_k) + (\bar{x}_k - m_k)^T W_k (\bar{x}_k - m_k) \}$$

$$- \text{Tr}(S_k W_k) + (z_n - \bar{z}_k)^T W_k (z_n - \bar{z}_k)$$

$$+ 2 (\bar{x}_k - m_k)^T W_k (z_n - \bar{x}_k) \}$$

$$(10.61) \text{ 例)} \\ S_k = \frac{1}{N_k} \sum_{n=1}^{N_k} (z_n - \bar{z}_k)(z_n - \bar{z}_k)^T \\ \frac{2}{N_k} \sum_{n=1}^{N_k} \frac{\partial}{\partial r_{nk}} r_{nk} (z_n - \bar{z}_k)(z_n - \bar{z}_k)^T \\ + \frac{2}{N_k} \sum_{n=1}^{N_k} \frac{\partial}{\partial r_{nk}} r_{nk} (z_n - \bar{z}_k)(z_n - \bar{z}_k)^T \\ = \frac{1}{N_k} S_k + \frac{1}{N_k} (z_n - \bar{z}_k)(z_n - \bar{z}_k)^T \\ = \frac{1}{N_k} \{ -S_k + (z_n - \bar{z}_k)(z_n - \bar{z}_k)^T \}$$

$$(10.62) \text{ 例)} \\ \bar{x}_k = \frac{1}{N_k} \sum_{n=1}^{N_k} z_n \\ \frac{2}{N_k} \bar{x}_k = \frac{1}{N_k} \sum_{n=1}^{N_k} z_n + \frac{1}{N_k} z_n \\ = \frac{1}{N_k} \bar{x}_k + \frac{1}{N_k} z_n \\ = \frac{1}{N_k} (z_n - \bar{x}_k)$$

$$\begin{aligned} & \frac{2}{N_k} (z_n - \bar{z}_k)(z_n - \bar{z}_k)^T \\ &= \frac{2}{N_k} (z_n - \bar{z}_k) \{ (z_n - \bar{z}_k)^T \\ &+ (z_n - \bar{z}_k) \frac{2}{N_k} (z_n - \bar{z}_k) \} \\ &= \frac{1}{N_k} (2z_n - \bar{z}_k)(z_n - \bar{z}_k)^T \\ &+ (z_n - \bar{z}_k) \frac{1}{N_k} (z_n - \bar{z}_k)^T \\ &= \frac{1}{N_k} \{ (z_n - \bar{z}_k)(z_n - \bar{z}_k)^T + (z_n - \bar{z}_k)(z_n - \bar{z}_k)^T \} \end{aligned}$$

$$\begin{aligned} & \frac{2}{N_k} \frac{2}{N_k} r_{nk} (z_n - \bar{z}_k)(z_n - \bar{z}_k)^T \\ &= \frac{2}{N_k} \{ r_{nk} (z_n - \bar{z}_k)(z_n - \bar{z}_k)^T + \frac{2}{N_k} r_{nk} (\bar{z}_k - \bar{z}_k)(z_n - \bar{z}_k)^T \} \\ &= (z_n - \bar{z}_k)(z_n - \bar{z}_k)^T + r_{nk} \frac{2}{N_k} (z_n - \bar{z}_k)(z_n - \bar{z}_k)^T \\ &+ \frac{2}{N_k} r_{nk} \frac{2}{N_k} (\bar{z}_k - \bar{z}_k)(z_n - \bar{z}_k)^T \\ &= (z_n - \bar{z}_k)(z_n - \bar{z}_k)^T + \frac{2}{N_k} r_{nk} \frac{2}{N_k} (\bar{z}_k - \bar{z}_k)(z_n - \bar{z}_k)^T \\ &= (z_n - \bar{z}_k)(z_n - \bar{z}_k)^T + \frac{2}{N_k} \frac{2}{N_k} r_{nk} \{ (z_n - \bar{z}_k)(z_n - \bar{z}_k)^T + (\bar{z}_k - \bar{z}_k)(z_n - \bar{z}_k)^T \} \\ &= (z_n - \bar{z}_k)(z_n - \bar{z}_k)^T \end{aligned}$$

$$\begin{aligned} & \frac{2}{N_k} r_{nk} (z_n - \bar{z}_k)(z_n - \bar{z}_k)^T \\ &= (z_n - \bar{z}_k) \{ \frac{2}{N_k} r_{nk} (\bar{z}_k - \bar{z}_k)^T \} \\ &= (z_n - \bar{z}_k) (\bar{z}_k - \bar{z}_k)^T \\ &= 0 \end{aligned}$$

$$\begin{aligned} & \frac{2}{N_k} (z_n - \bar{z}_k)^T W_k (z_n - \bar{z}_k) \\ &= \frac{2}{N_k} (z_n - \bar{z}_k)^T W_k (z_n - \bar{z}_k) \frac{2}{N_k} \\ &= 2 (\bar{x}_k - m_k)^T W_k \frac{2}{N_k} \\ &= \frac{2}{N_k} (\bar{x}_k - m_k)^T W_k (z_n - \bar{x}_k) \end{aligned}$$

$$- (\ln r_{nk} + 1) - \frac{D}{2} \ln(2\pi) + \lambda$$

$$= \frac{1}{2} \ln \tilde{\lambda}_k + \ln \tilde{\pi}_k - \frac{D}{2} \beta_k^{-1}$$

$$- \frac{1}{2} V_k \left(\bar{x}_k^T W_k \bar{x}_k - \bar{x}_k^T W_k m_k - m_k^T W_k \bar{x}_k + m_k^T W_k m_k \right)$$

$$+ z_n^T W_k z_n - z_n^T W_k \bar{x}_k - \bar{x}_k^T W_k z_n + \bar{x}_k^T W_k \bar{x}_k$$

$$+ 2 \bar{x}_k^T W_k z_n - 2 \bar{x}_k^T W_k \bar{x}_k - 2 m_k^T W_k z_n + 2 m_k^T W_k \bar{x}_k \right)$$

$$- (\ln r_{nk} + 1) - \frac{D}{2} \ln(2\pi) + \lambda$$

$$\begin{aligned}
&= \frac{1}{2} \ln \tilde{\lambda}_k + \ln \tilde{\pi}_k - \frac{D}{2} \beta_k^{-1} \\
&\quad - \frac{1}{2} V_k (x_n^T W_k x_n - 2m_k^T W_k x_n + m_k^T W_k m_k) \\
&\quad - (\ln r_n k + 1) - \frac{D}{2} \ln(2\pi) + \lambda \\
&= \frac{1}{2} \ln \tilde{\lambda}_k + \ln \tilde{\pi}_k - \frac{D}{2} \beta_k^{-1} \\
&\quad - \frac{1}{2} V_k (x_n - m_k)^T W_k (x_n - m_k) \\
&\quad - (\ln r_n k + 1) - \frac{D}{2} \ln(2\pi) + \lambda
\end{aligned}$$

∴

$$\ln r_n k = \frac{1}{2} \ln \tilde{\lambda}_k + \ln \tilde{\pi}_k - \frac{D}{2} \beta_k^{-1} - \frac{1}{2} V_k (x_n - m_k)^T W_k (x_n - m_k) - 1 - \frac{D}{2} \ln(2\pi) + \lambda$$

∴ (HFI)

$$r_n k = \tilde{\lambda}_k \tilde{\pi}_k \exp \left\{ -\frac{D}{2\beta_k} - \frac{V_k}{2} (x_n - m_k)^T W_k (x_n - m_k) \right\} \exp \left\{ -1 - \frac{D}{2} \ln(2\pi) + \lambda \right\}$$

∴

$$r_n k \propto \tilde{\lambda}_k \tilde{\pi}_k \exp \left\{ -\frac{D}{2\beta_k} - \frac{V_k}{2} (x_n - m_k)^T W_k (x_n - m_k) \right\}$$

を得る。