

(10.80) F1)

$$\begin{aligned} p(\hat{x}|X) &\propto \sum_k \int \pi_k N(\hat{x}|\mu_k, \Lambda_k^{-1}) g(\pi) g(\mu_k, \Lambda_k) d\pi d\mu_k d\Lambda_k \\ &= \sum_k \int \pi_k g(\pi) d\pi \int N(\hat{x}|\mu_k, \Lambda_k^{-1}) g(\mu_k, \Lambda_k) d\mu_k d\Lambda_k \\ &= \sum_k \int \pi_k \text{Dir}(\pi|\alpha) d\pi \int N(\hat{x}|\mu_k, \Lambda_k^{-1}) N(\mu_k|\mu_k, (\beta_k \Lambda_k^{-1})') W(\Lambda_k|W_k, V_k) d\mu_k d\Lambda_k \end{aligned}$$

ここで

$$\begin{aligned} \int \pi_k g(\pi) d\pi &= \int \pi_k \text{Dir}(\pi|\alpha) d\pi \quad \leftarrow (10.57) F1) \\ &= \frac{\alpha_k}{\alpha} \quad \leftarrow (B.17) F1) \end{aligned}$$

では

$$\begin{aligned} &\int N(\hat{x}|\mu_k, \Lambda_k^{-1}) g(\mu_k, \Lambda_k) d\mu_k d\Lambda_k \\ &= \int N(\hat{x}|\mu_k, \Lambda_k^{-1}) N(\mu_k|\mu_k, (\beta_k \Lambda_k^{-1})') W(\Lambda_k|W_k, V_k) d\mu_k d\Lambda_k \quad \leftarrow (10.59) F1) \\ &= \int W(\Lambda_k|W_k, V_k) \int N(\hat{x}|\mu_k, \Lambda_k^{-1}) N(\mu_k|\mu_k, (\beta_k \Lambda_k^{-1})') d\mu_k d\Lambda_k \\ &= \int W(\Lambda_k|W_k, V_k) N(\hat{x}|\mu_k, (1+\beta_k^{-1}) \Lambda_k^{-1}) d\Lambda_k \quad \leftarrow N(\hat{x}|\mu_k, \Lambda_k^{-1}) N(\mu_k|\mu_k, (\beta_k \Lambda_k^{-1})') d\mu_k \text{ は } N(\hat{x}|\mu_k, (1+\beta_k^{-1}) \Lambda_k^{-1}) \text{ である。} \\ &= \int B(W_k, V_k) |\Lambda_k|^{\frac{V_k-D-1}{2}} \exp\left\{-\frac{1}{2} \text{Tr}(W_k^{-1} \Lambda_k)\right\} \frac{1}{(2\pi)^{\frac{D}{2}}} \frac{1}{|(1+\beta_k^{-1}) \Lambda_k^{-1}|^{\frac{D}{2}}} \exp\left\{-\frac{1}{2} (\hat{x}-\mu_k)^T (1+\beta_k^{-1}) \Lambda_k (\hat{x}-\mu_k)\right\} d\Lambda_k \\ &= \frac{1}{(2\pi)^{\frac{D}{2}}} \frac{1}{(1+\beta_k^{-1})^{\frac{D}{2}}} \int B(W_k, V_k) |\Lambda_k|^{\frac{V_k-D}{2}} \exp\left[-\frac{1}{2} \left\{ \text{Tr}(W_k^{-1} \Lambda_k) + \text{Tr}\left((1+\beta_k^{-1})^{-1} (\hat{x}-\mu_k)(\hat{x}-\mu_k)^T \Lambda_k\right) \right\}\right] d\Lambda_k \\ &= \frac{1}{(2\pi)^{\frac{D}{2}}} \frac{1}{(1+\beta_k^{-1})^{\frac{D}{2}}} B(W_k, V_k) \int |\Lambda_k|^{\frac{V_k-D}{2}} \exp\left[-\frac{1}{2} \text{Tr}\left\{W_k^{-1} + (1+\beta_k^{-1})^{-1} (\hat{x}-\mu_k)(\hat{x}-\mu_k)^T\right\} \Lambda_k\right] d\Lambda_k \\ &= \frac{1}{(2\pi)^{\frac{D}{2}}} \frac{1}{(1+\beta_k^{-1})^{\frac{D}{2}}} B(W_k, V_k) \int \frac{1}{B(M_k, V_k+1)} W(\Lambda_k|M_k, V_k+1) d\Lambda_k, \quad M_k^{-1} = W_k^{-1} + (1+\beta_k^{-1})^{-1} (\hat{x}-\mu_k)(\hat{x}-\mu_k)^T \\ &= \frac{1}{(2\pi)^{\frac{D}{2}}} \frac{1}{(1+\beta_k^{-1})^{\frac{D}{2}}} \frac{B(W_k, V_k)}{B(M_k, V_k+1)} \\ &= \frac{1}{(2\pi)^{\frac{D}{2}}} \frac{1}{(1+\beta_k^{-1})^{\frac{D}{2}}} |W_k|^{\frac{D}{2}} \{1 + (1+\beta_k^{-1})^{-1} (\hat{x}-\mu_k)^T W_k (\hat{x}-\mu_k)\}^{-\frac{D}{2}} \frac{1}{2} \frac{\Gamma(\frac{V_k+1}{2})}{\Gamma(\frac{V_k+D}{2})} \left\{ \begin{aligned} &= \frac{B(W_k, V_k)}{B(M_k, V_k+1)} = \frac{|W_k|^{\frac{D}{2}} (2^{\frac{D}{2}} \pi^{\frac{D(D+1)}{4}} \frac{\Gamma(\frac{V_k+1}{2})}{\Gamma(\frac{V_k+D}{2})})^{-1}}{|M_k|^{\frac{D}{2}} (2^{\frac{D}{2}} \pi^{\frac{D(D+1)}{4}} \frac{\Gamma(\frac{V_k+1}{2})}{\Gamma(\frac{V_k+D}{2})})^{-1}} \\ &= |W_k|^{\frac{D}{2}} \{1 + (1+\beta_k^{-1})^{-1} (\hat{x}-\mu_k)^T W_k (\hat{x}-\mu_k)\}^{-\frac{D}{2}} \frac{1}{2} \frac{\Gamma(\frac{V_k+1}{2})}{\Gamma(\frac{V_k+D}{2})} \\ &= \frac{|W_k|^{\frac{D}{2}}}{|M_k|^{\frac{D}{2}}} = \frac{|W_k|^{\frac{D}{2}}}{|W_k^{-1} + (1+\beta_k^{-1})^{-1} (\hat{x}-\mu_k)(\hat{x}-\mu_k)^T|^{\frac{D}{2}}} \\ &= |W_k|^{\frac{D}{2}} |W_k^{-1} \{I + W_k (1+\beta_k^{-1})^{-1} (\hat{x}-\mu_k)(\hat{x}-\mu_k)^T\}|^{-\frac{D}{2}} \\ &= |W_k|^{\frac{D}{2}} |I + W_k (1+\beta_k^{-1})^{-1} (\hat{x}-\mu_k)(\hat{x}-\mu_k)^T|^{\frac{D}{2}} \quad \leftarrow (C.15) \\ &= |W_k|^{\frac{D}{2}} \{1 + (1+\beta_k^{-1})^{-1} (\hat{x}-\mu_k)^T W_k (\hat{x}-\mu_k)\}^{-\frac{D}{2}} \\ &= |W_k|^{\frac{D}{2}} \{1 + (1+\beta_k^{-1})^{-1} (\hat{x}-\mu_k)^T W_k (\hat{x}-\mu_k)\}^{-\frac{D}{2}} \end{aligned} \right. \end{aligned}$$

$$L_k = \frac{V_k+1-D}{1+\beta_k^{-1}} W_k$$

$$\Delta^2 = (\hat{x}-\mu_k)^T L_k (\hat{x}-\mu_k)$$

よって

$$p(\hat{x}|X) \propto \sum_k \frac{\alpha_k}{\alpha} \text{St}(\hat{x}|\mu_k, L_k, V_k+1-D) \dots (10.81)$$

を得る。