11.1

$\mathbf{Z} = (z^{(1)}, z^{(2)}, \cdots z^{(L)})$ の確率分布は、$z^{(l)}$ が独立なので

$p(\mathbf{Z}) = p(z^{(1)})\, p(z^{(2)}) \cdots$

これより

$$E[\hat{f}] = \int \hat{f}\, p(\mathbf{Z})\, d\mathbf{Z} = \int \frac{1}{L} \sum_{l=1}^{L} f(z^{(l)})\, p(z^{(1)})\, p(z^{(2)}) \cdots dz^{(1)} dz^{(2)} \cdots$$

$$= \frac{1}{L} \left\{ \int f(z^{(1)})\, p(z^{(1)}) p(z^{(2)}) \cdots dz^{(1)} dz^{(2)} \cdots + \int f(z^{(2)})\, p(z^{(1)}) p(z^{(2)}) \cdots dz^{(1)} dz^{(2)} \cdots \right\}$$

$$= \frac{1}{L} \left\{ \int f(z^{(1)})\, p(z^{(1)})\, dz^{(1)} + \int f(z^{(2)})\, p(z^{(2)})\, dz^{(2)} + \cdots \right\}$$

$$= \frac{1}{L} \sum_{l=1}^{L} \int f(z^{(l)})\, p(z^{(l)})\, dz^{(l)} = \frac{1}{L} \sum_{l=1}^{L} E[f] = E[f]$$

を得る。次に分散を求める。

$$\mathrm{var}[\hat{f}] = E[(\hat{f} - E[\hat{f}])^2] = E[\hat{f}^2 - 2\hat{f} E[\hat{f}] + (E[\hat{f}])^2]$$

$$= E[\hat{f}^2] - 2E[\hat{f}]E[\hat{f}] + (E[\hat{f}])^2$$

$E[\hat{f}] = E[f]$

$$= E[\hat{f}^2] - (E[\hat{f}])^2 = E[\hat{f}^2] - (E[f])^2$$

ここで

$$E[\hat{f}^2] = \int \hat{f}^2 p(\mathbf{Z})\, d\mathbf{Z} = \int \left(\frac{1}{L} \sum_{l=1}^{L} f(z^{(l)}) \right)^2 p(z^{(1)})\, p(z^{(2)}) \cdots dz^{(1)} dz^{(2)} \cdots$$

$$= \frac{1}{L^2} \int \left\{ \sum_{l=1}^{L} (f(z^{(l)}))^2 + \sum_{m \neq n} \sum_{n} f(z^{(m)}) f(z^{(n)}) \right\} p(z^{(1)})\, p(z^{(2)}) \cdots dz^{(1)} dz^{(2)} \cdots$$

$$= \frac{1}{L^2} \left[\int \sum_{l=1}^{L} (f(z^{(l)}))^2 p(z^{(1)})\, p(z^{(2)}) \cdots dz^{(1)} dz^{(2)} \cdots + \int \sum_{m \neq n} \sum_{n} f(z^{(m)}) f(z^{(n)})\, p(z^{(1)})\, p(z^{(2)}) \cdots dz^{(1)} dz^{(2)} \cdots \right]$$

$$= \frac{1}{L^2} \left[\sum_{l=1}^{L} \int (f(z^{(l)}))^2 p(z^{(l)})\, dz^{(l)} + \sum_{m \neq n} \sum_{n} \int f(z^{(m)}) f(z^{(n)})\, p(z^{(m)})\, p(z^{(n)})\, dz^{(m)} dz^{(n)} \right]$$

$$= \frac{1}{L^2} \left[\sum_{l=1}^{L} \int (f(z^{(l)}))^2 p(z^{(l)})\, dz^{(l)} + \sum_{m \neq n} \sum_{n} \int f(z^{(m)})\, p(z^{(m)})\, dz^{(m)} \int f(z^{(n)})\, p(z^{(n)})\, dz^{(n)} \right]$$

$$= \frac{1}{L^2} \left\{ L\, E[f^2] + L(L-1)(E[f])^2 \right\} = \frac{1}{L} E[f^2] + \frac{L(L-1)}{L^2} (E[f])^2$$

これより

$$\mathrm{var}[\hat{f}] = E[\hat{f}^2] - (E[f])^2$$

$$= \frac{1}{L}E[f^2] + \frac{L(L-1)}{L^2}(E[f])^2 - (E[f])^2$$

$$= \frac{1}{L}\{E[f^2] - (E[f])^2\}$$

$$= \frac{1}{L}E[(f-E[f])^2]$$

を得る