11.4

(11.10), (11.11)

$$y_1 = z_1\left(\frac{-2\ln r^2}{r^2}\right)^{1/2}$$

$$y_2 = z_2\left(\frac{-2\ln r^2}{r^2}\right)^{1/2}$$

$$r^2 = z_1^2 + z^2$$

これより

$$y_1^2 + y_2^2 = (z_1^2 + z_2^2)\frac{-2\ln r^2}{r^2} = -2\ln r^2$$

$$\therefore\ r^2 = \exp\left\{-\frac{1}{2}(y_1^2+y_2^2)\right\}$$

これらを (11.10) に戻して

$$y_1 = z_1\left[\frac{y_1^2+y_2^2}{\exp\{-\frac{1}{2}(y_1^2+y_2^2)\}}\right]^{\frac{1}{2}}$$

$$\therefore\ z_1 = y_1\left[\frac{\exp\{-\frac{1}{2}(y_1^2+y_2^2)\}}{y_1^2+y_2^2}\right]^{\frac{1}{2}}$$

同様に (11.11) から

$$z_2 = y_2\left[\frac{\exp\{-\frac{1}{2}(y_1^2+y_2^2)\}}{y_1^2+y_2^2}\right]^{\frac{1}{2}}$$

$$\frac{\partial}{\partial y_1}\frac{\exp\{-\frac{1}{2}(y_1^2+y_2^2)\}}{y_1^2+y_2^2}$$

$$= \frac{-y_1\exp\{-\frac{1}{2}(y_1^2+y_2^2)\}(y_1^2+y_2^2) - \exp\{-\frac{1}{2}(y_1^2+y_2^2)\}\cdot 2y_1}{(y_1^2+y_2^2)^2}$$

$$= \frac{\exp\{-\frac{1}{2}(y_1^2+y_2^2)\}}{(y_1^2+y_2^2)^2}(-y_1)(y_1^2+y_2^2+2)$$

これより

$$\frac{\partial z_1}{\partial y_1} = \left[\frac{\exp\{-\frac{1}{2}(y_1^2+y_2^2)\}}{y_1^2+y_2^2}\right]^{\frac{1}{2}}$$

$$+ y_1\frac{1}{2}\left[\frac{\exp\{-\frac{1}{2}(y_1^2+y_2^2)\}}{y_1^2+y_2^2}\right]^{-\frac{1}{2}}\frac{\exp\{-\frac{1}{2}(y_1^2+y_2^2)\}}{(y_1^2+y_2^2)^2}(-y_1)(y_1^2+y_2^2+2)$$

$$= \left[\frac{\exp\{-\frac{1}{2}(y_1^2+y_2^2)\}}{y_1^2+y_2^2}\right]^{\frac{1}{2}}\left(1 - \frac{y_1^2}{2}\frac{y_1^2+y_2^2+2}{y_1^2+y_2^2}\right)$$

$$\frac{\partial z_1}{\partial y_2} = y_1 \frac{1}{2}\left[\frac{\exp\{-\frac{1}{2}(y_1^2+y_2^2)\}}{y_1^2+y_2^2}\right]^{-\frac{1}{2}} \frac{\exp\{-\frac{1}{2}(y_1^2+y_2^2)\}}{(y_1^2+y_2^2)^2}(-y_2)(y_1^2+y_2^2+2)$$

$$= -\frac{y_1 y_2}{2}\left[\frac{\exp\{-\frac{1}{2}(y_1^2+y_2^2)\}}{y_1^2+y_2^2}\right]^{\frac{1}{2}} \frac{y_1^2+y_2^2+2}{y_1^2+y_2^2}$$

同様に

$$\frac{\partial z_2}{\partial y_2} = \left[\frac{\exp\{-\frac{1}{2}(y_1^2+y_2^2)\}}{y_1^2+y_2^2}\right]^{\frac{1}{2}}\left(1-\frac{y_2^2}{2}\frac{y_1^2+y_2^2+2}{y_1^2+y_2^2}\right)$$

$$\frac{\partial z_2}{\partial y_1} = -\frac{y_1 y_2}{2}\left[\frac{\exp\{-\frac{1}{2}(y_1^2+y_2^2)\}}{y_1^2+y_2^2}\right]^{\frac{1}{2}} \frac{y_1^2+y_2^2+2}{y_1^2+y_2^2}$$

$p(y_1, y_2)$ の確率密度は、(11.9) より

$$p(y_1, y_2) = p(z_1, z_2)\left|\det\left(\frac{\partial(z_1,z_2)}{\partial(y_1,y_2)}\right)\right|$$

$$= p(z_1, z_2)\left|\det\begin{pmatrix}\frac{\partial z_1}{\partial y_1} & \frac{\partial z_1}{\partial y_2} \\ \frac{\partial z_2}{\partial y_1} & \frac{\partial z_2}{\partial y_2}\end{pmatrix}\right|$$

$$= p(z_1, z_2)\left|\frac{\partial z_1}{\partial y_1}\frac{\partial z_2}{\partial y_2} - \frac{\partial z_1}{\partial y_2}\frac{\partial z_2}{\partial y_1}\right|$$

ここで

$$\frac{\partial z_1}{\partial y_1}\frac{\partial z_2}{\partial y_2} - \frac{\partial z_1}{\partial y_2}\frac{\partial z_2}{\partial y_1} = \left[\frac{\exp\{-\frac{1}{2}(y_1^2+y_2^2)\}}{y_1^2+y_2^2}\right]^{\frac{1}{2}}\left(1-\frac{y_1^2}{2}\frac{y_1^2+y_2^2+2}{y_1^2+y_2^2}\right)\left[\frac{\exp\{-\frac{1}{2}(y_1^2+y_2^2)\}}{y_1^2+y_2^2}\right]^{\frac{1}{2}}\left(1-\frac{y_2^2}{2}\frac{y_1^2+y_2^2+2}{y_1^2+y_2^2}\right)$$

$$-\left[-\frac{y_1 y_2}{2}\left[\frac{\exp\{-\frac{1}{2}(y_1^2+y_2^2)\}}{y_1^2+y_2^2}\right]^{\frac{1}{2}}\frac{y_1^2+y_2^2+2}{y_1^2+y_2^2}\right]\left[-\frac{y_1 y_2}{2}\left[\frac{\exp\{-\frac{1}{2}(y_1^2+y_2^2)\}}{y_1^2+y_2^2}\right]^{\frac{1}{2}}\frac{y_1^2+y_2^2+2}{y_1^2+y_2^2}\right]$$

$$= \frac{\exp\{-\frac{1}{2}(y_1^2+y_2^2)\}}{y_1^2+y_2^2}\left(1-\frac{y_1^2}{2}\frac{y_1^2+y_2^2+2}{y_1^2+y_2^2}\right)\left(1-\frac{y_2^2}{2}\frac{y_1^2+y_2^2+2}{y_1^2+y_2^2}\right)$$

$$-\frac{y_1^2 y_2^2}{4}\frac{\exp\{-\frac{1}{2}(y_1^2+y_2^2)\}}{y_1^2+y_2^2}\left(\frac{y_1^2+y_2^2+2}{y_1^2+y_2^2}\right)^2$$

$$= \frac{\exp\{-\frac{1}{2}(y_1^2+y_2^2)\}}{y_1^2+y_2^2}\left\{1-\frac{y_1^2+y_2^2+2}{2}+\frac{y_1^2 y_2^2}{4}\left(\frac{y_1^2+y_2^2+2}{y_1^2+y_2^2}\right)^2-\frac{y_1^2 y_2^2}{4}\left(\frac{y_1^2+y_2^2+2}{y_1^2+y_2^2}\right)^2\right\}$$

$$= \frac{\exp\{-\frac{1}{2}(y_1^2+y_2^2)\}}{y_1^2+y_2^2}\left(1-\frac{y_1^2+y_2^2+2}{2}\right)$$

$$= -\frac{1}{2}\exp\{-\frac{1}{2}(y_1^2+y_2^2)\}$$

これより

$$p(y_1, y_2) = p(z_1, z_2)\left|\frac{\partial z_1}{\partial y_1}\frac{\partial z_2}{\partial y_2} - \frac{\partial z_1}{\partial y_2}\frac{\partial z_2}{\partial y_1}\right|$$

$$= \frac{1}{2\pi}\exp\left\{-\frac{1}{2}(y_1^2+y_2^2)\right\} \longleftarrow p(z_1, z_2) = \frac{1}{\pi}$$

$$= \frac{1}{\sqrt{2\pi}}\exp\left(-\frac{1}{2}y_1^2\right)\frac{1}{\sqrt{2\pi}}\exp\left(-\frac{1}{2}y_2^2\right)$$

を得る。