13.10

$$p(\mathbf{X},\mathbf{Z}) = p(x_1 \cdots x_N, z_1 \cdots z_N) = p(z_1)\left[\prod_{i=2}^{N} p(z_i|z_{i-1})\right]\prod_{i=1}^{N} p(x_i|z_i) \quad \cdots \quad (13.6)$$

((13.24) の導出)

(13.6) より

$$p(x_1 \cdots x_n, z_n) = \sum_{x_{n+1}} \cdots \sum_{x_N} p(\mathbf{X}, z_n) = \sum_{x_{n+1}} \cdots \sum_{x_N} \sum_{\mathbf{Z}\setminus n} p(\mathbf{X},\mathbf{Z})$$

(← 加法定理, ← 加法定理)

$$= \sum_{x_{n+1}} \cdots \sum_{x_N} \sum_{\mathbf{Z}\setminus n} p(z_1)\left[\prod_{i=2}^{N} p(z_i|z_{i-1})\right]\prod_{i=1}^{N} p(x_i|z_i)$$

(← (13.6))

$$= \sum_{\mathbf{Z}\setminus n} \sum_{x_{n+1}} \cdots \sum_{x_N} p(z_1)\left[\prod_{i=2}^{N} p(z_i|z_{i-1})\right]\prod_{i=1}^{N} p(x_i|z_i)$$

← 和の順番は交換できる

$$\sum_a\sum_b ab = \sum_a a(b_1+b_2) = a_1b_1+a_1b_2+a_2b_1+a_2b_2 = \sum_b\sum_a ab$$

$$= \sum_{\mathbf{Z}\setminus n} p(z_1)\left[\prod_{i=2}^{N} p(z_i|z_{i-1})\right]\prod_{i=1}^{n} p(x_i|z_i) \sum_{x_{n+1}} \cdots \sum_{x_N} \prod_{i=n+1}^{N} p(x_i|z_i)$$

← $x_{n+1} \cdots x_N$ に依らない項は $\sum_{x_{n+1}} \cdots \sum_{x_N}$ からくくり出せる

$$\sum_a\sum_b\sum_c abc = \sum_a\sum_b abc_1+abc_2 = \sum_a\sum_b ab\left(\sum_c c\right)$$

$$= \sum_{\mathbf{Z}\setminus n} p(z_1)\left[\prod_{i=2}^{N} p(z_i|z_{i-1})\right]\prod_{i=1}^{n} p(x_i|z_i) \prod_{i=n+1}^{N} \underbrace{\sum_{x_i} p(x_i|z_i)}_{=1}$$

← x_i の項は x_j の項と混ざってないので積と和は入れ替えて和の積にできる

$$(例)\ \sum_a\sum_b\sum_c abc = \sum_a\sum_b (abc_1+abc_2) = \sum_a\sum_b ab(c_1+c_2) = \left(\sum_a\sum_b ab\right)\left(\sum_c c\right) = \left(\sum_a a\right)\left(\sum_b b\right)\left(\sum_c c\right)$$

$$= \sum_{\mathbf{Z}\setminus n} p(z_1)\left[\prod_{i=2}^{N} p(z_i|z_{i-1})\right]\prod_{i=1}^{n} p(x_i|z_i)$$

$$= \sum_{z_1} \cdots \sum_{z_{n-1}} \sum_{z_{n+1}} \cdots \sum_{z_N} p(z_1)\left[\prod_{i=2}^{n} p(z_i|z_{i-1})\right]\prod_{i=1}^{n} p(x_i|z_i)$$

$$= \sum_{z_1} \cdots \sum_{z_{n-1}} p(z_1)\left[\prod_{i=2}^{n} p(z_i|z_{i-1})\right]\prod_{i=1}^{n} p(x_i|z_i) \underbrace{\left(\sum_{z_{n+1}} \cdots \sum_{z_N}\left[\prod_{i=n+1}^{N} p(z_i|z_{i-1})\right]\right)}_{=1}$$

← $z_{n+1} \sim z_N$ を含まない項を $\sum_{z_{n+1}} \sim \sum_{z_N}$ の外にくくり出す

$$= \sum_{z_1} \cdots \sum_{z_{n-1}} p(z_1)\left[\prod_{i=2}^{n} p(z_i|z_{i-1})\right]\prod_{i=1}^{n} p(x_i|z_i)$$

となる。

$$\sum_{z_{n+1}} \cdots \sum_{z_N}\left[\prod_{i=n+1}^{N} p(z_i|z_{i-1})\right] = \sum_{z_{n+1}} \cdots \sum_{z_N} p(z_{n+1}|z_n) \cdots p(z_N|z_{N-1}) = \sum_{z_{n+1}} p(z_{n+1}|z_n) \cdots \underbrace{\sum_{z_N} p(z_N|z_{N-1})}_{=1} = 1$$

$\cdots = 1$

同様に

$$p(x_{n+1}\cdots x_N, z_n) = \sum_{x_1}\cdots\sum_{x_n}\sum_{\mathbf{Z}_{\backslash n}} p(\mathbf{X},\mathbf{Z})$$

$$= \sum_{x_1}\cdots\sum_{x_n}\sum_{\mathbf{Z}_{\backslash n}} p(z_1)\left[\prod_{i=2}^{N} p(z_i|z_{i-1})\right]\prod_{i=1}^{N} p(x_i|z_i)$$

$$= \sum_{\mathbf{Z}_{\backslash n}} p(z_1)\left[\prod_{i=2}^{N} p(z_i|z_{i-1})\right]\prod_{i=n+1}^{N} p(x_i|z_i)\underbrace{\sum_{x_1}\cdots\sum_{x_n}\prod_{i=1}^{n} p(x_i|z_i)}_{=1}$$

← $x_1 \sim x_n$ を含まない項を $\sum_{z_1}\sim\sum_{x_n}$ の外にくくり出す

$$\sum_{x_1}\cdots\sum_{x_n}\prod_{i=1}^{n} p(x_i|z_i) = \prod_{i=1}^{n}\underbrace{\sum_{x_i} p(x_i|z_i)}_{=1} = 1$$

← x_i の項が混ざってないので積と和を交換できる

$$= \sum_{\mathbf{Z}_{\backslash n}} p(z_1)\left[\prod_{i=2}^{N} p(z_i|z_{i-1})\right]\prod_{i=n+1}^{N} p(x_i|z_i)$$

$$= \sum_{z_{n+1}}\cdots\sum_{z_N}\left[\prod_{i=n+1}^{N} p(z_i|z_{i-1})\right]\prod_{i=n+1}^{N} p(x_i|z_i)\sum_{z_1}\cdots\sum_{z_{n-1}} p(z_1)\left[\prod_{i=2}^{n} p(z_i|z_{i-1})\right]$$

← $z_1 \sim z_{n-1}$ を含まない項を $\sum_{z_1}\cdots\sum_{z_{n-1}}$ の外にくくり出す

$$= \left(\sum_{z_{n+1}}\cdots\sum_{z_N}\left[\prod_{i=n+1}^{N} p(z_i|z_{i-1})\right]\prod_{i=n+1}^{N} p(x_i|z_i)\right)\left(\underbrace{\sum_{z_1}\cdots\sum_{z_{n-1}} p(z_1)\left[\prod_{i=2}^{n} p(z_i|z_{i-1})\right]}\right)$$

← この項は z_n のみの関数となり $z_{n+1}\sim z_N$ には依存しないので $\sum_{z_{n+1}}\cdots\sum_{z_N}$ の和の外にくくり出せる

となる。

ここで

$$p(z_n) = \sum_{\mathbf{X}}\sum_{\mathbf{Z}_{\backslash n}} p(\mathbf{X},\mathbf{Z}) = \sum_{\mathbf{X}}\sum_{\mathbf{Z}_{\backslash n}} p(z_1)\left[\prod_{i=2}^{N} p(z_i|z_{i-1})\right]\prod_{i=1}^{N} p(x_i|z_i)$$

$$= \sum_{\mathbf{Z}_{\backslash n}} p(z_1)\left[\prod_{i=2}^{N} p(z_i|z_{i-1})\right]\underbrace{\sum_{\mathbf{X}}\prod_{i=1}^{N} p(x_i|z_i)}_{=1}$$

← $\mathbf{X}$ に依らない項を $\sum_{\mathbf{X}}$ の外にくくり出す

← x_i の項がまざってないので積和の順番を交換できる

$$= \sum_{\mathbf{Z}_{\backslash n}} p(z_1)\left[\prod_{i=2}^{N} p(z_i|z_{i-1})\right]$$

$$= \sum_{z_1}\cdots\sum_{z_{n-1}}\sum_{z_{n+1}}\cdots\sum_{z_N} p(z_1)\left[\prod_{i=2}^{N} p(z_i|z_{i-1})\right]$$

← $z_{n+1}\cdots z_N$ を含まない項を $\sum_{z_{n+1}}\cdots\sum_{z_N}$ の外にくくり出す

$$= \sum_{z_1}\cdots\sum_{z_{n-1}} p(z_1)\left[\prod_{i=2}^{n} p(z_i|z_{i-1})\right]\underbrace{\sum_{z_{n+1}}\cdots\sum_{z_N}\left[\prod_{i=n+1}^{N} p(z_i|z_{i-1})\right]}_{=1}$$

$$\sum_{z_{n+1}}\cdots\sum_{z_N}\left[\prod_{i=n+1}^{N} p(z_i|z_{i-1})\right] = \sum_{z_{n+1}}\cdots\sum_{z_N} p(z_{n+1}|z_n)\cdots p(z_N|z_{N-1})$$

$$= \sum_{z_{n+1}} p(z_{n+1}|z_n)\cdots\underbrace{\sum_{z_N} p(z_N|z_{N-1})}_{\cdots\ =1} = 1$$

$$= \sum_{z_1}\cdots\sum_{z_{n-1}} p(z_1)\left[\prod_{i=2}^{n} p(z_i|z_{i-1})\right]$$

なので

$$p(x_{n+1}\cdots x_N, z_n) = \left(\sum_{z_{n+1}}\cdots\sum_{z_N}\left[\prod_{i=n+1}^{N} p(z_i|z_{i-1})\right]\prod_{i=n+1}^{N} p(x_i|z_i)\right)p(z_n)$$

$$\therefore\ p(x_{n+1}\cdots x_N | z_n) = \left(\sum_{z_{n+1}}\cdots\sum_{z_N}\left[\prod_{i=n+1}^{N} p(z_i|z_{i-1})\right]\prod_{i=n+1}^{N} p(x_i|z_i)\right)$$

となる。

よって

$$p(\mathbf{x}_1 \cdots \mathbf{x}_n, \mathbf{z}_n)\, p(\mathbf{x}_{n+1} \cdots \mathbf{x}_N \mid \mathbf{z}_n)$$

$$= \underbrace{\left(\sum_{\mathbf{z}_1} \cdots \sum_{\mathbf{z}_{n-1}} p(\mathbf{z}_1) \left[\prod_{i=2}^{n} p(\mathbf{z}_i \mid \mathbf{z}_{i-1}) \right] \prod_{i=1}^{n} p(\mathbf{x}_i \mid \mathbf{z}_i) \right)}_{\text{この項は } \mathbf{z}_{n+1} \sim \mathbf{z}_N \text{ を含まないので } \sum_{\mathbf{z}_{n+1}} \cdots \sum_{\mathbf{z}_N} \text{ の中に入れてよい}} \left(\sum_{\mathbf{z}_{n+1}} \cdots \sum_{\mathbf{z}_N} \left[\prod_{i=n+1}^{N} p(\mathbf{z}_i \mid \mathbf{z}_{i-1}) \right] \prod_{i=n+1}^{N} p(\mathbf{x}_i \mid \mathbf{z}_i) \right)$$

$$= \sum_{\mathbf{z}_{n+1}} \cdots \sum_{\mathbf{z}_N} \left(\sum_{\mathbf{z}_1} \cdots \sum_{\mathbf{z}_{n-1}} p(\mathbf{z}_1) \left[\prod_{i=2}^{n} p(\mathbf{z}_i \mid \mathbf{z}_{i-1}) \right] \prod_{i=1}^{n} p(\mathbf{x}_i \mid \mathbf{z}_i) \right) \underbrace{\left[\prod_{i=n+1}^{N} p(\mathbf{z}_i \mid \mathbf{z}_{i-1}) \right] \prod_{i=n+1}^{N} p(\mathbf{x}_i \mid \mathbf{z}_i)}_{\text{この項は } \mathbf{z}_1 \sim \mathbf{z}_{n-1} \text{ を含まないので } \sum_{\mathbf{z}_1} \cdots \sum_{\mathbf{z}_{n-1}} \text{ の中に入れてよい}}$$

$$= \sum_{\mathbf{z}_{n+1}} \cdots \sum_{\mathbf{z}_N} \left(\sum_{\mathbf{z}_1} \cdots \sum_{\mathbf{z}_{n-1}} p(\mathbf{z}_1) \left[\prod_{i=2}^{n} p(\mathbf{z}_i \mid \mathbf{z}_{i-1}) \right] \prod_{i=1}^{n} p(\mathbf{x}_i \mid \mathbf{z}_i) \left[\prod_{i=n+1}^{N} p(\mathbf{z}_i \mid \mathbf{z}_{i-1}) \right] \prod_{i=n+1}^{N} p(\mathbf{x}_i \mid \mathbf{z}_i) \right)$$

↙ 和の順番は変えてよい

$$= \sum_{\mathbf{z}_1} \cdots \sum_{\mathbf{z}_{n-1}} \sum_{\mathbf{z}_{n+1}} \cdots \sum_{\mathbf{z}_N} p(\mathbf{z}_1) \left[\prod_{i=2}^{N} p(\mathbf{z}_i \mid \mathbf{z}_{i-1}) \right] \prod_{i=1}^{N} p(\mathbf{x}_i \mid \mathbf{z}_i)$$

$$= \sum_{\mathbf{Z} \setminus n} p(\mathbf{X}, \mathbf{Z}) = p(\mathbf{X}, \mathbf{z}_n)$$

を得る。両辺を $p(\mathbf{z}_n)$ で割ると

$$\frac{p(\mathbf{x}_1 \cdots \mathbf{x}_n, \mathbf{z}_n)\, p(\mathbf{x}_{n+1} \cdots \mathbf{x}_N \mid \mathbf{z}_n)}{p(\mathbf{z}_n)} = \frac{p(\mathbf{X}, \mathbf{z}_n)}{p(\mathbf{z}_n)}$$

積の定理

$$\therefore\ p(\mathbf{x}_1 \cdots \mathbf{x}_n \mid \mathbf{z}_n)\, p(\mathbf{x}_{n+1} \cdots \mathbf{x}_N \mid \mathbf{z}_n) = p(\mathbf{X} \mid \mathbf{z}_n) \quad \cdots (13.24)$$

を得る。

（(13.25) の導出）

$$p(x_1 \cdots x_{n-1}, z_n) = \sum_{x_n} \cdots \sum_{x_N} \sum_{z_{\setminus n}} p(X, Z)$$

↖ 和の定理

$$= \sum_{x_n} \cdots \sum_{x_N} \sum_{z_{\setminus n}} p(z_1) \left[\prod_{i=2}^{N} p(z_i | z_{i-1}) \right] \prod_{i=1}^{N} p(x_i | z_i)$$

↖ (13.6)

x_n〜x_N を含まない項を $\sum_{x_n} \cdots \sum_{x_N}$ の外にくくり出す

$$= \sum_{z_{\setminus n}} p(z_1) \left[\prod_{i=2}^{N} p(z_i | z_{i-1}) \right] \prod_{i=1}^{n-1} p(x_i | z_i) \underbrace{\sum_{x_n} \cdots \sum_{x_N} \prod_{i=n}^{N} p(x_i | z_i)}_{=1}$$

← x_i の項は混ざっていないので
積和は和積に交換できる
また $\sum_{x_i} p(x_i | z_i) = 1$

$$= \sum_{z_1} \cdots \sum_{z_{n-1}} p(z_1) \left[\prod_{i=2}^{n} p(z_i | z_{i-1}) \right] \prod_{i=1}^{n-1} p(x_i | z_i) \underbrace{\sum_{z_{n+1}} \cdots \sum_{z_N} \left[\prod_{i=n+1}^{N} p(z_i | z_{i-1}) \right]}_{=1}$$

← z_{n+1}〜z_N 含まない項を
$\sum_{z_{n+1}} \cdots \sum_{z_N}$ の外にくくり出す

$$= \sum_{z_1} \cdots \sum_{z_{n-1}} p(z_1) \left[\prod_{i=2}^{n} p(z_i | z_{i-1}) \right] \prod_{i=1}^{n-1} p(x_i | z_i)$$

$$\sum_{z_{n+1}} \cdots \sum_{z_N} \left[\prod_{i=n+1}^{N} p(z_i | z_{i-1}) \right]$$

$$= \sum_{z_{n+1}} p(z_{n+1} | z_n) \cdots \underbrace{\sum_{z_N} p(z_N | z_{N-1})}_{=1}$$

$$= 1 \qquad \cdots$$

同様に

$$p(x_1 \cdots x_{n-1}, x_n, z_n) = \sum_{x_{n+1}} \cdots \sum_{x_N} \sum_{z_{\setminus n}} p(X, Z)$$

$$= \sum_{x_{n+1}} \cdots \sum_{x_N} \sum_{z_{\setminus n}} p(z_1) \left[\prod_{i=2}^{N} p(z_i | z_{i-1}) \right] \prod_{i=1}^{N} p(x_i | z_i)$$

$$= \sum_{z_{\setminus n}} p(z_1) \left[\prod_{i=2}^{N} p(z_i | z_{i-1}) \right] \prod_{i=1}^{n} p(x_i | z_i) \underbrace{\sum_{x_{n+1}} \cdots \sum_{x_N} \prod_{i=n+1}^{N} p(x_i | z_i)}_{=1}$$

$$= \sum_{z_{\setminus n}} p(z_1) \left[\prod_{i=2}^{N} p(z_i | z_{i-1}) \right] \prod_{i=1}^{n} p(x_i | z_i)$$

$$= \sum_{z_1} \cdots \sum_{z_{n-1}} p(z_1) \left[\prod_{i=2}^{n} p(z_i | z_{i-1}) \right] \prod_{i=1}^{n} p(x_i | z_i) \underbrace{\sum_{z_{n+1}} \cdots \sum_{z_N} \left[\prod_{i=n+1}^{N} p(z_i | z_{i-1}) \right]}_{=1}$$

$$= \sum_{z_1} \cdots \sum_{z_{n-1}} p(z_1) \left[\prod_{i=2}^{n} p(z_i | z_{i-1}) \right] \prod_{i=1}^{n} p(x_i | z_i)$$

$$= \left(\sum_{z_1} \cdots \sum_{z_{n-1}} p(z_1) \left[\prod_{i=2}^{n} p(z_i | z_{i-1}) \right] \prod_{i=1}^{n-1} p(x_i | z_i) \right) p(x_n | z_n)$$

よって

$$p(x_1 \cdots x_{n-1}, x_n, z_n) = p(x_1 \cdots x_{n-1}, z_n) \, p(x_n | z_n)$$

となる。両辺を $p(x_n, z_n) = p(x_n | z_n) p(z_n)$ で割ると

$$\frac{p(x_1 \cdots x_{n-1}, x_n, z_n)}{p(x_n, z_n)} = \frac{p(x_1 \cdots x_{n-1}, z_n) \, p(x_n | z_n)}{p(x_n | z_n) \, p(z_n)}$$

$$\therefore \; p(x_1 \cdots x_{n-1} | x_n, z_n) = p(x_1 \cdots x_{n-1} | z_n) \quad \cdots (13.25)$$

を得る

（（13.26）の導出）

$$p(x_1 \cdots x_{n-1}, z_{n-1}) = \sum_{x_n} \cdots \sum_{x_N} \sum_{Z \backslash n-1} p(X, Z)$$

（和の定理）

（13.6）

$$= \sum_{x_n} \cdots \sum_{x_N} \sum_{Z \backslash n-1} p(z_1) \left[\prod_{i=2}^{N} p(z_i | z_{i-1}) \right] \prod_{i=1}^{N} p(x_i | z_i)$$

（$x_n \cdots x_N$ を含まない項を $\sum_{x_n} \cdots \sum_{x_N}$ の外にくくり出す）

$$= \sum_{Z \backslash n-1} p(z_1) \left[\prod_{i=2}^{N} p(z_i | z_{i-1}) \right] \prod_{i=1}^{n-1} p(x_i | z_i) \underbrace{\sum_{x_n} \cdots \sum_{x_N} \prod_{i=n}^{N} p(x_i | z_i)}_{=1}$$

（x_i の項がすべて独立で積と和を交換できる。また $\sum_{x_i} p(x_i | z_i) = 1$）

$$= \sum_{Z \backslash n-1} p(z_1) \left[\prod_{i=2}^{N} p(z_i | z_{i-1}) \right] \prod_{i=1}^{n-1} p(x_i | z_i)$$

（$z_n \cdots z_N$ を含まない項を $\sum_{z_n} \cdots \sum_{z_N}$ の外にくくり出す）

$$= \sum_{z_1} \cdots \sum_{z_{n-2}} p(z_1) \left[\prod_{i=2}^{n-1} p(z_i | z_{i-1}) \right] \prod_{i=1}^{n-1} p(x_i | z_i) \underbrace{\sum_{z_n} \cdots \sum_{z_N} \left[\prod_{i=n}^{N} p(z_i | z_{i-1}) \right]}_{=1}$$

$$\sum_{z_n} \cdots \sum_{z_N} \left[\prod_{i=n}^{N} p(z_i | z_{i-1}) \right] = \sum_{z_n} \cdots \sum_{z_N} p(z_n | z_{n-1}) \cdots p(z_N | z_{N-1}) = \sum_{z_n} p(z_n | z_{n-1}) \cdots \underbrace{\sum_{z_N} p(z_N | z_{N-1})}_{=1} = 1$$

$$= \sum_{z_1} \cdots \sum_{z_{n-2}} p(z_1) \left[\prod_{i=2}^{n-1} p(z_i | z_{i-1}) \right] \prod_{i=1}^{n-1} p(x_i | z_i)$$

同じように

$$p(x_1 \cdots x_{n-1}, z_{n-1}, z_n) = \sum_{x_n} \cdots \sum_{x_N} \sum_{z_1} \cdots \sum_{z_{n-2}} \sum_{z_{n+1}} \cdots \sum_{z_N} p(X, Z)$$

$$= \sum_{x_n} \cdots \sum_{x_N} \sum_{z_1} \cdots \sum_{z_{n-2}} \sum_{z_{n+1}} \cdots \sum_{z_N} p(z_1) \left[\prod_{i=2}^{N} p(z_i | z_{i-1}) \right] \prod_{i=1}^{N} p(x_i | z_i)$$

$$= \sum_{z_1} \cdots \sum_{z_{n-2}} \sum_{z_{n+1}} \cdots \sum_{z_N} p(z_1) \left[\prod_{i=2}^{N} p(z_i | z_{i-1}) \right] \prod_{i=1}^{n-1} p(x_i | z_i) \underbrace{\sum_{x_n} \cdots \sum_{x_N} \prod_{i=n}^{N} p(x_i | z_i)}_{=1}$$

$$= \sum_{z_1} \cdots \sum_{z_{n-2}} \sum_{z_{n+1}} \cdots \sum_{z_N} p(z_1) \left[\prod_{i=2}^{N} p(z_i | z_{i-1}) \right] \prod_{i=1}^{n-1} p(x_i | z_i)$$

$$= \sum_{z_1} \cdots \sum_{z_{n-2}} p(z_1) \left[\prod_{i=2}^{n} p(z_i | z_{i-1}) \right] \prod_{i=1}^{n-1} p(x_i | z_i) \underbrace{\sum_{z_{n+1}} \cdots \sum_{z_N} \left[\prod_{i=n+1}^{N} p(z_i | z_{i-1}) \right]}_{=1}$$

$$= \sum_{z_1} \cdots \sum_{z_{n-2}} p(z_1) \left[\prod_{i=2}^{n} p(z_i | z_{i-1}) \right] \prod_{i=1}^{n-1} p(x_i | z_i)$$

$$= \left(\sum_{z_1} \cdots \sum_{z_{n-2}} p(z_1) \left[\prod_{i=2}^{n-1} p(z_i | z_{i-1}) \right] \prod_{i=1}^{n-1} p(x_i | z_i) \right) p(z_n | z_{n-1})$$

（上の $p(x_1 \cdots x_{n-1}, z_{n-1})$ になっている）

よって

$$p(x_1 \cdots x_{n-1}, z_{n-1}, z_n) = p(x_1 \cdots x_{n-1}, z_{n-1}) \, p(z_n | z_{n-1})$$

となる。両辺を $p(z_{n-1}, z_n) = p(z_n | z_{n-1}) p(z_{n-1})$ で割ると

$$\therefore \frac{p(x_1 \cdots x_{n-1}, z_{n-1}, z_n)}{p(z_{n-1}, z_n)} = \frac{p(x_1 \cdots x_{n-1}, z_{n-1}) \, p(z_n | z_{n-1})}{p(z_n | z_{n-1}) \, p(z_{n-1})}$$

$$\therefore p(x_1 \cdots x_{n-1} | z_{n-1}, z_n) = p(x_1 \cdots x_{n-1} | z_{n-1}) \quad \cdots (13.26)$$

を得る。

((13.27) の導出)

$$p(x_{n+1} \cdots x_N, z_{n+1}) = \sum_{x_1} \cdots \sum_{x_n} \sum_{z_{\setminus n+1}} p(X, Z)$$ ← 和の定理

(13.6)

$$= \sum_{x_1} \cdots \sum_{x_n} \sum_{z_{\setminus n+1}} p(z_1) \left[\prod_{i=2}^{N} p(z_i | z_{i-1}) \right] \prod_{i=1}^{N} p(x_i | z_i)$$

$x_1 \cdots x_n$ を含まない項を $\sum_{x_1} \cdots \sum_{x_n}$ の外にくくり出す

$$= \sum_{z_{\setminus n+1}} p(z_1) \left[\prod_{i=2}^{N} p(z_i | z_{i-1}) \right] \prod_{i=n+1}^{N} p(x_i | z_i) \underbrace{\sum_{x_1} \cdots \sum_{x_n} \prod_{i=1}^{n} p(x_i | z_i)}_{=1}$$

x_i の項が混ざってないので積と和を交換できる また $\sum_{x_i} p(x_i | z_i) = 1$

$$= \sum_{z_{\setminus n+1}} p(z_1) \left[\prod_{i=2}^{N} p(z_i | z_{i-1}) \right] \prod_{i=n+1}^{N} p(x_i | z_i)$$

$z_1 \cdots z_n$ を含まない項を $\sum_{z_1} \cdots \sum_{z_n}$ の外にくくり出す

$$= \sum_{z_{n+2}} \cdots \sum_{z_N} \left[\prod_{i=n+2}^{N} p(z_i | z_{i-1}) \right] \prod_{i=n+1}^{N} p(x_i | z_i) \sum_{z_1} \cdots \sum_{z_n} p(z_1) \left[\prod_{i=2}^{n+1} p(z_i | z_{i-1}) \right]$$

$$= \left(\sum_{z_{n+2}} \cdots \sum_{z_N} \left[\prod_{i=n+2}^{N} p(z_i | z_{i-1}) \right] \prod_{i=n+1}^{N} p(x_i | z_i) \right) \left(\underline{\sum_{z_1} \cdots \sum_{z_n} p(z_1) \left[\prod_{i=2}^{n+1} p(z_i | z_{i-1}) \right]} \right)$$

この項は z_{n+1} のみの関数なので $\sum_{z_{n+2}} \cdots \sum_{z_N}$ の外に出せる

ここで

$$p(z_{n+1}) = \sum_{X} \sum_{Z_{\setminus n+1}} p(X, Z) = \sum_{X} \sum_{z_{\setminus n+1}} p(z_1) \left[\prod_{i=2}^{N} p(z_i | z_{i-1}) \right] \prod_{i=1}^{N} p(x_i | z_i)$$

上と同じで X を含まない項くくり出す

$$= \sum_{z_{\setminus n+1}} p(z_1) \left[\prod_{i=2}^{N} p(z_i | z_{i-1}) \right] \underbrace{\sum_{X} \prod_{i=1}^{N} p(x_i | z_i)}_{=1}$$

$z_{n+2} \cdots z_N$ に依らない項を $\sum_{z_{n+2}} \cdots \sum_{z_N}$ からくくり出す

$$= \sum_{z_1} \cdots \sum_{z_n} p(z_1) \left[\prod_{i=2}^{n+1} p(z_i | z_{i-1}) \right] \underbrace{\sum_{z_{n+2}} \cdots \sum_{z_N} \left[\prod_{i=n+2}^{N} p(z_i | z_{i-1}) \right]}_{=1}$$

$$\sum_{z_{n+2}} p(z_{n+2} | z_{n+1}) \cdots \underbrace{\sum_{z_N} p(z_N | z_{N-1})}_{\cdots = 1} = 1$$

$$= \sum_{z_1} \cdots \sum_{z_n} p(z_1) \left[\prod_{i=2}^{n+1} p(z_i | z_{i-1}) \right]$$

なので

$$p(x_{n+1} \cdots x_N, z_{n+1}) = \left(\sum_{z_{n+2}} \cdots \sum_{z_N} \left[\prod_{i=n+2}^{N} p(z_i | z_{i-1}) \right] \prod_{i=n+1}^{N} p(x_i | z_i) \right) p(z_{n+1})$$

$$\therefore p(x_{n+1} \cdots x_N | z_{n+1}) = \sum_{z_{n+2}} \cdots \sum_{z_N} \left[\prod_{i=n+2}^{N} p(z_i | z_{i-1}) \right] \prod_{i=n+1}^{N} p(x_i | z_i)$$

↑ 積の定理

となる。

同じように

$$p(x_{n+1}\cdots x_N, z_n, z_{n+1}) = \sum_{x_1}\cdots\sum_{x_n}\sum_{z_1}\cdots\sum_{z_{n-1}}\sum_{z_{n+2}}\cdots\sum_{z_N} p(\mathbf{X},\mathbf{Z})$$ ← 和の定理

$$= \sum_{x_1}\cdots\sum_{x_n}\sum_{z_1}\cdots\sum_{z_{n-1}}\sum_{z_{n+2}}\cdots\sum_{z_N} p(z_1)\left[\prod_{i=2}^{N} p(z_i|z_{i-1})\right]\prod_{i=1}^{N} p(x_i|z_i)$$ ←(13.6)

$x_1\cdots x_n$ を含まない項を $\sum_{x_1}\cdots\sum_{x_n}$ の外にくくり出す

$$= \sum_{z_1}\cdots\sum_{z_{n-1}}\sum_{z_{n+2}}\cdots\sum_{z_N} p(z_1)\left[\prod_{i=2}^{N} p(z_i|z_{i-1})\right]\prod_{i=n+1}^{N} p(x_i|z_i)\underbrace{\sum_{x_1}\cdots\sum_{x_n}\prod_{i=1}^{n} p(x_i|z_i)}_{=1}$$

x_i の項が混ざってないので積と和を交換できる　また $\sum_{x_i} p(x_i|z_i)=1$

$$= \sum_{z_1}\cdots\sum_{z_{n-1}}\sum_{z_{n+2}}\cdots\sum_{z_N} p(z_1)\left[\prod_{i=2}^{N} p(z_i|z_{i-1})\right]\prod_{i=n+1}^{N} p(x_i|z_i)$$

$z_1\cdots z_{n-1}$ を含まない項を $\sum_{z_1}\cdots\sum_{z_{n-1}}$ の外にくくり出す

$$= \sum_{z_{n+2}}\cdots\sum_{z_N}\left[\prod_{i=n+1}^{N} p(z_i|z_{i-1})\right]\prod_{i=n+1}^{N} p(x_i|z_i)\sum_{z_1}\cdots\sum_{z_{n-1}} p(z_1)\left[\prod_{i=2}^{n} p(z_i|z_{i-1})\right]$$

この項は z_n のみの関数となり $\sum_{z_{n+2}}\cdots\sum_{z_N}$ の外に出せる

$$= \left(\sum_{z_{n+2}}\cdots\sum_{z_N}\left[\prod_{i=n+1}^{N} p(z_i|z_{i-1})\right]\prod_{i=n+1}^{N} p(x_i|z_i)\right)\left(\sum_{z_1}\cdots\sum_{z_{n-1}} p(z_1)\left[\prod_{i=2}^{n} p(z_i|z_{i-1})\right]\right)$$

$$= p(z_{n+1}|z_n)\left(\sum_{z_{n+2}}\cdots\sum_{z_N}\left[\prod_{i=n+2}^{N} p(z_i|z_{i-1})\right]\prod_{i=n+1}^{N} p(x_i|z_i)\right)\left(\sum_{z_1}\cdots\sum_{z_{n-1}} p(z_1)\left[\prod_{i=2}^{n} p(z_i|z_{i-1})\right]\right)$$

ここで

$$p(z_n) = \sum_{\mathbf{X}}\sum_{\mathbf{Z}\backslash n} p(\mathbf{X},\mathbf{Z}) = \sum_{\mathbf{X}}\sum_{\mathbf{Z}\backslash n} p(z_1)\left[\prod_{i=2}^{N} p(z_i|z_{i-1})\right]\prod_{i=1}^{N} p(x_i|z_i)$$

$$= \sum_{\mathbf{Z}\backslash n} p(z_1)\left[\prod_{i=2}^{N} p(z_i|z_{i-1})\right]\underbrace{\sum_{\mathbf{X}}\prod_{i=1}^{N} p(x_i|z_i)}_{=1}$$

$$= \sum_{z_1}\cdots\sum_{z_{n-1}} p(z_1)\left[\prod_{i=2}^{n} p(z_i|z_{i-1})\right]\underbrace{\sum_{z_{n+1}}\cdots\sum_{z_N}\left[\prod_{i=n+1}^{N} p(z_i|z_{i-1})\right]}_{=1}$$

なので

$$p(x_{n+1}\cdots x_N, z_n, z_{n+1}) = p(z_{n+1}|z_n)\left(\sum_{z_{n+2}}\cdots\sum_{z_N}\left[\prod_{i=n+2}^{N} p(z_i|z_{i-1})\right]\prod_{i=n+1}^{N} p(x_i|z_i)\right)p(z_n)$$

積の定理

$$\therefore\ p(x_{n+1}\cdots x_N | z_n, z_{n+1}) = \sum_{z_{n+2}}\cdots\sum_{z_N}\left[\prod_{i=n+2}^{N} p(z_i|z_{i-1})\right]\prod_{i=n+1}^{N} p(x_i|z_i)$$

← $p(x_{n+1}\cdots x_N | z_{n+1})$ と同じになっている

となる。

よって

$$p(x_{n+1}\cdots x_N | z_n, z_{n+1}) = p(x_{n+1}\cdots x_N | z_{n+1}) \quad \cdots (13.27)$$

を得る。

(（13.28）の導出）

$$p(x_{n+2} \cdots x_N, z_{n+1}) = \sum_{x_1} \cdots \sum_{x_{n+1}} \sum_{z_{n+1}} p(X, Z)$$

← 和の定理

$$= \sum_{x_1} \cdots \sum_{x_{n+1}} \sum_{z_{n+1}} p(z_1) \left[\prod_{i=2}^{N} p(z_i | z_{i-1}) \right] \prod_{i=1}^{N} p(x_i | z_i)$$

← (13.6)

> $x_1 \cdots x_{n+1}$ に依らない項を $\sum_{x_1} \cdots \sum_{x_{n+1}}$ の外に出す

$$= \sum_{z_{n+1}} p(z_1) \left[\prod_{i=2}^{N} p(z_i | z_{i-1}) \right] \prod_{i=n+2}^{N} p(x_i | z_i) \underbrace{\sum_{x_1} \cdots \sum_{x_{n+1}} \prod_{i=1}^{n+1} p(x_i | z_i)}_{=1}$$

> x_i の項が分離できるので和積を交換できる $\sum_{x_i} p(x_i | z_i) = 1$ である

$$= \sum_{z_{n+1}} p(z_1) \left[\prod_{i=2}^{N} p(z_i | z_{i-1}) \right] \prod_{i=n+2}^{N} p(x_i | z_i)$$

> $z_1 \sim z_n$ に依らない項を $\sum_{z_1} \cdots \sum_{z_n}$ の外に出す

$$= \sum_{z_{n+2}} \cdots \sum_{z_N} \left[\prod_{i=n+2}^{N} p(z_i | z_{i-1}) \right] \prod_{i=n+2}^{N} p(x_i | z_i) \sum_{z_1} \cdots \sum_{z_n} p(z_1) \left[\prod_{i=2}^{n+1} p(z_i | z_{i-1}) \right]$$

> この項は z_{n+1} だけの関数なので $\sum_{z_{n+2}} \cdots \sum_{z_N}$ の外に出せる

$$= \left(\sum_{z_{n+2}} \cdots \sum_{z_N} \left[\prod_{i=n+2}^{N} p(z_i | z_{i-1}) \right] \prod_{i=n+2}^{N} p(x_i | z_i) \right) \left(\sum_{z_1} \cdots \sum_{z_n} p(z_1) \left[\prod_{i=2}^{n+1} p(z_i | z_{i-1}) \right] \right)$$

ここで

$$p(z_{n+1}) = \sum_{X} \sum_{Z \backslash n+1} p(X, Z) = \sum_{X} \sum_{Z \backslash n+1} p(z_1) \left[\prod_{i=2}^{N} p(z_i | z_{i-1}) \right] \prod_{i=1}^{N} p(x_i | z_i)$$

$$= \sum_{z_1} \cdots \sum_{z_n} \sum_{z_{n+2}} \cdots \sum_{z_N} p(z_1) \left[\prod_{i=2}^{N} p(z_i | z_{i-1}) \right] \underbrace{\sum_{X} \prod_{i=1}^{N} p(x_i | z_i)}_{=1}$$

$$= \sum_{z_1} \cdots \sum_{z_n} p(z_1) \left[\prod_{i=2}^{n+1} p(z_i | z_{i-1}) \right] \underbrace{\sum_{z_{n+2}} \cdots \sum_{z_N} \left[\prod_{i=n+2}^{N} p(z_i | z_{i-1}) \right]}_{=1}$$

$$= \sum_{z_1} \cdots \sum_{z_n} p(z_1) \left[\prod_{i=2}^{n+1} p(z_i | z_{i-1}) \right]$$

なので

$$p(x_{n+2} \cdots x_N, z_{n+1}) = \left(\sum_{z_{n+2}} \cdots \sum_{z_N} \left[\prod_{i=n+2}^{N} p(z_i | z_{i-1}) \right] \prod_{i=n+2}^{N} p(x_i | z_i) \right) p(z_{n+1})$$

$$\therefore p(x_{n+2} \cdots x_N | z_{n+1}) = \left(\sum_{z_{n+2}} \cdots \sum_{z_N} \left[\prod_{i=n+2}^{N} p(z_i | z_{i-1}) \right] \prod_{i=n+2}^{N} p(x_i | z_i) \right)$$

↑ 積の定理

となる。

同じように

$$p(x_{n+2} \cdots x_N, z_{n+1}, x_{n+1}) = \sum_{x_1}\cdots\sum_{x_n}\sum_{z_{1 \sim n+1}} p(X, Z)$$ ← 和の定理

$$= \sum_{x_1}\cdots\sum_{x_n}\sum_{z_{1 \sim n+1}} p(z_1)\left[\prod_{i=2}^{N} p(z_i|z_{i-1})\right]\prod_{i=1}^{N} p(x_i|z_i)$$ ←(13.6)

$x_1 \sim x_n$ に依らない項を $\sum_{x_1}\cdots\sum_{x_n}$ の外に出す

$$= \sum_{z_{1 \sim n+1}} p(z_1)\left[\prod_{i=2}^{N} p(z_i|z_{i-1})\right]\prod_{i=n+1}^{N} p(x_i|z_i)\underbrace{\sum_{x_1}\cdots\sum_{x_n}\prod_{i=1}^{n} p(x_i|z_i)}_{=1}$$

x_i の項が混ざってない また $\sum_{x_i} p(x_i|z_i)=1$

$z_1 \sim z_n$ に依らない項を $\sum_{z_1}\cdots\sum_{z_n}$ の外に出す

$$= \sum_{z_{n+2}}\cdots\sum_{z_N}\left[\prod_{i=n+2}^{N} p(z_i|z_{i-1})\right]\prod_{i=n+1}^{N} p(x_i|z_i)\sum_{z_1}\cdots\sum_{z_n} p(z_1)\left[\prod_{i=2}^{n+1} p(z_i|z_{i-1})\right]$$

この項は z_{n+1} のみの関数なので $\sum_{z_{n+2}}\cdots\sum_{z_N}$ の外に出せる

$$= \left(\sum_{z_{n+2}}\cdots\sum_{z_N}\left[\prod_{i=n+2}^{N} p(z_i|z_{i-1})\right]\prod_{i=n+1}^{N} p(x_i|z_i)\right)\underbrace{\left(\sum_{z_1}\cdots\sum_{z_n} p(z_1)\left[\prod_{i=2}^{n+1} p(z_i|z_{i-1})\right]\right)}_{=p(z_{n+1})}$$

← この上の $p(x_{n+2} \cdots x_N, z_{n+1})$ で求めたのと同じ

$$= \left(\sum_{z_{n+2}}\cdots\sum_{z_N}\left[\prod_{i=n+2}^{N} p(z_i|z_{i-1})\right]\prod_{i=n+1}^{N} p(x_i|z_i)\right) p(z_{n+1})$$

$$= p(x_{n+1}|z_{n+1})\left(\sum_{z_{n+2}}\cdots\sum_{z_N}\left[\prod_{i=n+2}^{N} p(z_i|z_{i-1})\right]\prod_{i=n+2}^{N} p(x_i|z_i)\right) p(z_{n+1})$$

よって

積の定理 ↓ ; $p(x_{n+2} \cdots x_N | z_{n+1})$ の右辺と同じ

$$\therefore p(x_{n+2} \cdots x_N | z_{n+1}, x_{n+1}) = \left(\sum_{z_{n+2}}\cdots\sum_{z_N}\left[\prod_{i=n+2}^{N} p(z_i|z_{i-1})\right]\prod_{i=n+2}^{N} p(x_i|z_i)\right)$$

となる。

よって

$$p(x_{n+2} \cdots x_N | z_{n+1}, x_{n+1}) = p(x_{n+2} \cdots x_N | z_{n+1}) \quad \cdots \quad (13.28)$$

を得る。

((13.29) の導出)

(13.24) の導出の途中で求めたように

$$p(x_{n+1}\cdots x_N \mid z_n) = \sum_{z_{n+1}}\cdots\sum_{z_N}\left[\prod_{i=n+1}^{N} p(z_i \mid z_{i-1})\right]\prod_{i=n+1}^{N} p(x_i \mid z_i)$$

(13.26) の導出の途中で求めたように

$$p(x_1\cdots x_{n-1}, z_{n-1}) = \sum_{z_1}\cdots\sum_{z_{n-2}} p(z_1)\left[\prod_{i=2}^{n-1} p(z_i \mid z_{i-1})\right]\prod_{i=1}^{n-1} p(x_i \mid z_i)$$

これより

$$p(x_1\cdots x_{n-1}, z_{n-1})\,p(x_{n+1}\cdots x_N \mid z_n)\,p(x_n \mid z_n)\,p(z_n \mid z_{n-1})$$

この項は $z_{n+1}\cdots z_N$ を含まないので $\sum_{z_{n+1}}\cdots\sum_{z_N}$ の中に入れてよい

$$= \left(\sum_{z_1}\cdots\sum_{z_{n-2}} p(z_1)\left[\prod_{i=2}^{n-1} p(z_i \mid z_{i-1})\right]\prod_{i=1}^{n-1} p(x_i \mid z_i)\right)\left(\sum_{z_{n+1}}\cdots\sum_{z_N}\left[\prod_{i=n+1}^{N} p(z_i \mid z_{i-1})\right]\prod_{i=n+1}^{N} p(x_i \mid z_i)\right) p(x_n \mid z_n)\,p(z_n \mid z_{n-1})$$

$$= \left(\sum_{z_{n+1}}\cdots\sum_{z_N}\sum_{z_1}\cdots\sum_{z_{n-2}} p(z_1)\left[\prod_{i=2}^{n-1} p(z_i \mid z_{i-1})\right]\prod_{i=1}^{n-1} p(x_i \mid z_i)\left[\prod_{i=n+1}^{N} p(z_i \mid z_{i-1})\right]\prod_{i=n+1}^{N} p(x_i \mid z_i)\right) p(x_n \mid z_n)\,p(z_n \mid z_{n-1})$$

$z_1\cdots z_{n-2}\ z_{n+1}\cdots z_N$ を含まないので $\sum_{z_{n+1}}\cdots\sum_{z_N}\sum_{z_1}\cdots\sum_{z_{n-2}}$ に入れてよい

$$= \sum_{z_{n+1}}\cdots\sum_{z_N}\sum_{z_1}\cdots\sum_{z_{n-2}} p(z_1)\left[\prod_{i=2}^{n-1} p(z_i \mid z_{i-1})\right]\prod_{i=1}^{n-1} p(x_i \mid z_i)\left[\prod_{i=n+1}^{N} p(z_i \mid z_{i-1})\right]\prod_{i=n+1}^{N} p(x_i \mid z_i)\ p(x_n \mid z_n)\,p(z_n \mid z_{n-1})$$

和の順番は入れかえてよい

$$= \sum_{z_1}\cdots\sum_{z_{n-2}}\sum_{z_{n+1}}\cdots\sum_{z_N} p(z_1)\left[\prod_{i=2}^{N} p(z_i \mid z_{i-1})\right]\prod_{i=1}^{N} p(x_i \mid z_i)$$

$$= p(X, z_{n-1}, z_n)$$

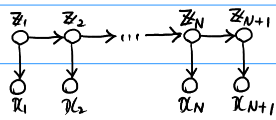
となる。両辺を $p(z_{n-1}, z_n) = p(z_n \mid z_{n-1})\,p(z_{n-1})$ で割ると

$$\frac{p(x_1\cdots x_{n-1}, z_{n-1})\,p(x_{n+1}\cdots x_N \mid z_n)\,p(x_n \mid z_n)\,p(z_n \mid z_{n-1})}{p(z_n \mid z_{n-1})\,p(z_{n-1})} = \frac{p(X, z_{n-1}, z_n)}{p(z_{n-1}, z_n)}$$

$$\therefore\ p(x_1\cdots x_{n-1} \mid z_{n-1})\,p(x_{n+1}\cdots x_N \mid z_n)\,p(x_n \mid z_n) = p(X \mid z_{n-1}, z_n) \quad\cdots(13.29)$$

を得る。

（(13.30) の導出）



モデル図より $x_1 \cdots x_{N+1}$, $z_1 \cdots z_{N+1}$ の同時分布は

$$p(\mathbf{X}, x_{N+1}, \mathbf{Z}, z_{N+1}) = p(\mathbf{X}, \mathbf{Z})\, p(z_{N+1} | z_N)\, p(x_{N+1} | z_{N+1})$$

これより

和の定理

$$p(x_{N+1}, \mathbf{X}, z_{N+1}) = \sum_{\mathbf{Z}} p(\mathbf{X}, x_{N+1}, \mathbf{Z}, z_{N+1})$$

$$= \sum_{\mathbf{Z}} p(\mathbf{X}, \mathbf{Z})\, p(z_{N+1} | z_N)\, p(x_{N+1} | z_{N+1})$$

$$= \underbrace{\left(\sum_{\mathbf{Z}} p(\mathbf{X}, \mathbf{Z})\, p(z_{N+1} | z_N) \right)}_{p(\mathbf{X}, z_{N+1}) \text{ になっている}} p(x_{N+1} | z_{N+1})$$

$z_1 \cdots z_N$ を含まないので和の外に出せる

ここで

$$p(\mathbf{X}, z_{N+1}) = \sum_{x_{N+1}} \sum_{\mathbf{Z}} p(\mathbf{X}, x_{N+1}, \mathbf{Z}, z_{N+1})$$

$$= \sum_{x_{N+1}} \sum_{\mathbf{Z}} p(\mathbf{X}, \mathbf{Z})\, p(z_{N+1} | z_N)\, p(x_{N+1} | z_{N+1})$$

x_{N+1} に依らない項を和の外に出した

$$= \sum_{\mathbf{Z}} p(\mathbf{X}, \mathbf{Z})\, p(z_{N+1} | z_N) \underbrace{\sum_{x_{N+1}} p(x_{N+1} | z_{N+1})}_{=1}$$

$$= \sum_{\mathbf{Z}} p(\mathbf{X}, \mathbf{Z})\, p(z_{N+1} | z_N)$$

なので

$$p(x_{N+1}, \mathbf{X}, z_{N+1}) = p(\mathbf{X}, z_{N+1})\, p(x_{N+1} | z_{N+1})$$

積の定理より

$$\therefore\ p(x_{N+1} | \mathbf{X}, z_{N+1}) = p(x_{N+1} | z_{N+1}) \quad \cdots \quad (13.30)$$

を得る。

((13.31) の導出)

(13.30) の導出と同じく $x_1 \cdots x_{N+1}$, $z_1 \cdots z_{N+1}$ の同時分布は

$$p(X, x_{N+1}, Z, z_{N+1}) = p(X, Z)\, p(z_{N+1}|z_N)\, p(x_{N+1}|z_{N+1})$$

これより

$$p(z_{N+1}, z_N, X) = \sum_{x_{N+1}} \sum_{z_1} \cdots \sum_{z_{N-1}} p(X, x_{N+1}, Z, z_{N+1})$$

$$= \sum_{x_{N+1}} \sum_{z_1} \cdots \sum_{z_{N-1}} p(X, Z)\, p(z_{N+1}|z_N)\, p(x_{N+1}|z_{N+1})$$

x_{N+1} に依らない項を $\sum_{x_{N+1}}$ の外に出した

$$= \sum_{z_1} \cdots \sum_{z_{N-1}} p(X, Z)\, p(z_{N+1}|z_N) \underbrace{\sum_{x_{N+1}} p(x_{N+1}|z_{N+1})}_{=1}$$

$$= \underbrace{\left(\sum_{z_1} \cdots \sum_{z_{N-1}} p(X, Z) \right)}_{= p(z_N, X)} p(z_{N+1}|z_N)$$

← この項は $z_1 \cdots z_{N-1}$ を含まないので和の外に出せる

ここで

$$p(z_N, X) = \sum_{x_{N+1}} \sum_{z_1} \cdots \sum_{z_{N-1}} \sum_{z_{N+1}} p(X, x_{N+1}, Z, z_{N+1})$$

$$= \sum_{x_{N+1}} \sum_{z_1} \cdots \sum_{z_{N-1}} \sum_{z_{N+1}} p(X, Z)\, p(z_{N+1}|z_N)\, p(x_{N+1}|z_{N+1})$$

$$= \sum_{z_1} \cdots \sum_{z_{N-1}} \sum_{z_{N+1}} p(X, Z)\, p(z_{N+1}|z_N) \underbrace{\sum_{x_{N+1}} p(x_{N+1}|z_{N+1})}_{=1}$$

z_{N+1} に依らない項を $\sum_{z_{N+1}}$ の外に出す

$$= \sum_{z_1} \cdots \sum_{z_{N-1}} p(X, Z) \underbrace{\sum_{z_{N+1}} p(z_{N+1}|z_N)}_{=1}$$

$$= \sum_{z_1} \cdots \sum_{z_{N-1}} p(X, Z)$$

よって

$$p(z_{N+1}, z_N, X) = p(z_N, X)\, p(z_{N+1}|z_N)$$

$$\therefore\ p(z_{N+1}|z_N, X) = p(z_{N+1}|z_N) \quad \cdots \quad (13.31)$$

を得る。