13.16

(13.6) より

$$p(x_1 \cdots x_{n+1}, z_1 \cdots z_{n+1}) = p(z_1)\left[\prod_{i=2}^{n+2} p(z_i|z_{i-1})\right]\left[\prod_{i=1}^{n+2} p(x_i|z_i)\right]$$

両辺の対数を取って最大化すると

$$\max_{z_1 \cdots z_{n+1}} p(x_1 \cdots x_{n+1}, z_1 \cdots z_{n+1}) = \max_{z_1 \cdots z_{n+1}} \left( \ln p(z_1) + \sum_{i=2}^{n+1} \ln p(z_i|z_{i-1}) + \sum_{i=1}^{n+1} \ln p(x_i|z_i) \right)$$

$$= \max_{z_{n+1}} \max_{z_1 \cdots z_n} \left( \ln p(z_1) + \sum_{i=2}^{n+1} \ln p(z_i|z_{i-1}) + \sum_{i=1}^{n+1} \ln p(x_i|z_i) \right)$$

$z_1 \cdots z_n$ を含まない項は $\max_{z_1 \cdots z_n}$ の外に出せる

$$= \max_{z_{n+1}} \left( \ln p(x_{n+1}|z_{n+1}) + \max_{z_1 \cdots z_n} \left( \ln p(z_1) + \sum_{i=2}^{n+1} \ln p(z_i|z_{i-1}) + \sum_{i=1}^{n} \ln p(x_i|z_i) \right) \right) \cdots ①$$

$$= \max_{z_{n+1}} \left( \ln p(x_{n+1}|z_{n+1}) + \max_{z_n} \max_{z_1 \cdots z_{n-1}} \left( \ln p(z_1) + \sum_{i=2}^{n+1} \ln p(z_i|z_{i-1}) + \sum_{i=1}^{n} \ln p(x_i|z_i) \right) \right)$$

$z_1 \cdots z_{n-1}$ を含まない項を $\max_{z_1 \cdots z_{n-1}}$ の外に出す

$$= \max_{z_{n+1}} \left( \ln p(x_{n+1}|z_{n+1}) + \max_{z_n} \left( \ln p(z_{n+1}|z_n) + \ln p(x_n|z_n) + \max_{z_1 \cdots z_{n-1}} \left( \ln p(z_1) + \sum_{i=2}^{n} \ln p(z_i|z_{i-1}) + \sum_{i=1}^{n-1} \ln p(x_i|z_i) \right) \right) \right) \cdots ②$$

ここで、$n > 1$ について

$$\omega(z_n) \equiv \ln p(x_n|z_n) + \max_{z_1 \cdots z_{n-1}} \left( \ln p(z_1) + \sum_{i=2}^{n} \ln p(z_i|z_{i-1}) + \sum_{i=1}^{n-1} \ln p(x_i|z_i) \right)$$

とすると、①, ② より

$$\max_{z_{n+1}} \left( \omega(z_{n+1}) \right) = \max_{z_{n+1}} \left( \ln p(x_{n+1}|z_{n+1}) + \max_{z_n} \left( \ln p(z_{n+1}|z_n) + \omega(z_n) \right) \right)$$

これより

$$\omega(z_{n+1}) = \ln p(x_{n+1}|z_{n+1}) + \max_{z_n} \left( \ln p(z_{n+1}|z_n) + \omega(z_n) \right) \cdots (13.68)$$

を得る。

さらに $\max\limits_{z_1 \cdots z_{n+1}} p(x_1 \cdots x_{n+1}, z_1 \cdots z_{n+1})$ の計算を進めると

$$\max_{z_1 \cdots z_{n+1}} p(x_1 \cdots x_{n+1}, z_1 \cdots z_{n+1})$$

$$= \max_{z_{n+1}} \left( \ln p(x_{n+1} | z_{n+1}) + \max_{z_n} \left( \ln p(z_{n+1} | z_n) + \ln p(x_n | z_n) + \max_{z_1 \cdots z_{n-1}} \left( \ln p(z_1) + \sum_{i=2}^{n} \ln p(z_i | z_{i-1}) + \sum_{i=1}^{n-1} \ln p(x_i | z_i) \right) \right) \right)$$

$$\vdots$$

$$= \max_{z_{n+1}} \left( \ln p(x_{n+1} | z_{n+1}) + \max_{z_n} \left( \cdots + \max_{z_1, z_2} \left( \ln p(z_1) + \sum_{i=2}^{3} \ln p(z_i | z_{i-1}) + \sum_{i=1}^{2} \ln p(x_i | z_i) \right) \right) \right)$$

$$= \max_{z_{n+1}} \left( \ln p(x_{n+1} | z_{n+1}) + \max_{z_n} \left( \cdots + \max_{z_2} \left( \ln p(z_3 | z_2) + \ln p(x_2 | z_2) + \max_{z_1} \left( \ln p(z_1) + \ln p(z_2 | z_1) + \ln p(x_1 | z_1) \right) \right) \cdots \right) \right)$$

これと，(13.68)で $n=2$ としたときを見比べて

$$\omega(z_2) = \ln p(x_2 | z_2) + \max_{z_1} \left( \ln p(z_2 | z_1) + \omega(z_1) \right)$$

$n=1$ のとき

$$\omega(z_1) \equiv \ln p(z_1) + \ln p(x_1 | z_1) \quad \cdots (13.69)$$

と定義する。