7.16

(7.100) より

$$\frac{d\lambda(\alpha_i)}{d\alpha_i} = \frac{\alpha_i^{-1}s_i^2 - (g_i^2 - s_i)}{2(\alpha_i + s_i)^2}$$

これより

$$\frac{d^2\lambda}{d\alpha_i^2} = \frac{1}{2}\frac{(-1)\alpha_i^{-2}s_i^2(\alpha_i+s_i)^2 - 2(\alpha_i+s_i)\{\alpha_i^{-1}s_i^2 - (g_i^2-s_i)\}}{(\alpha_i+s_i)^4}$$

$$= \frac{1}{2}\frac{-\alpha_i^{-2}s_i^2(\alpha_i+s_i) - 2\{\alpha_i^{-1}s_i^2 - (g_i^2-s_i)\}}{(\alpha_i+s_i)^3}$$

$$= \frac{1}{2}\frac{-\alpha_i^{-1}s_i^2 - \alpha_i^{-2}s_i^3 - 2\alpha_i^{-1}s_i^2 + 2g_i^2 - 2s_i}{(\alpha_i+s_i)^3}$$

$$= \frac{1}{2}\frac{-\alpha_i^{-2}(s_i^3 + 3\alpha_i s_i^2 + 2\alpha_i^2 s_i) + 2g_i^2}{(\alpha_i+s_i)^3}$$

$$= \frac{1}{2}\frac{-s_i^3 - 3\alpha_i s_i^2 - 2\alpha_i^2 s_i + 2\alpha_i^2 g_i^2}{\alpha_i^2(\alpha_i+s_i)^3}$$

これの正負を見るのだが、面倒なので図7.11の左図の例 $g_i^2=4$, $s_i=1$ の場合を考えることとする

このとき

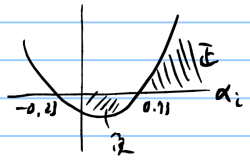
$$\frac{d^2\lambda}{d\alpha_i^2} = \frac{1}{2}\frac{-1-3\alpha_i-2\alpha_i^2+8\alpha_i^2}{\alpha_i^2(\alpha_i+1)^3} = \frac{1}{2}\frac{6\alpha_i^2-3\alpha_i-1}{\alpha_i^2(\alpha_i+1)^3}$$

となる。分母は正なので、分子の正負のみ考える

$$6\alpha_i^2 - 3\alpha_i - 1 = 0$$

より

$$\alpha_i = \frac{3\pm\sqrt{33}}{12} = -0.23,\ 0.73$$



なので $\lambda(\alpha_i)$ の2階微分は

$\lambda''(\alpha_i) < 0 \quad (0 \le \alpha_i < 0.73)$

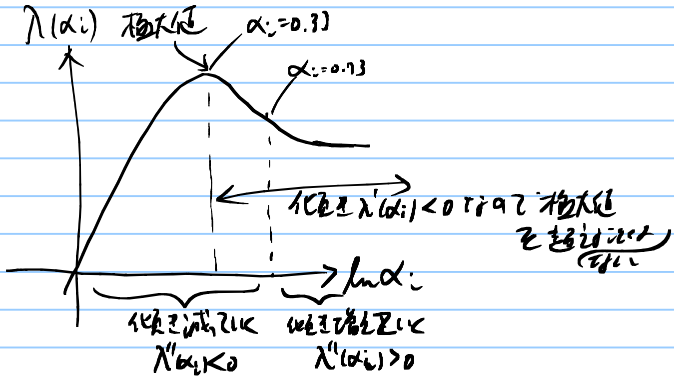
$\lambda''(\alpha_i) > 0 \quad (\alpha_i > 0.73)$

である。$\alpha_i = 0.73$ は変曲点となる

$\lambda(\alpha_i)$ の停留点は、(7.101) より

$$\alpha_i = \frac{1}{3} = 0.33$$

なので、この停留点は唯一の極大値となる。



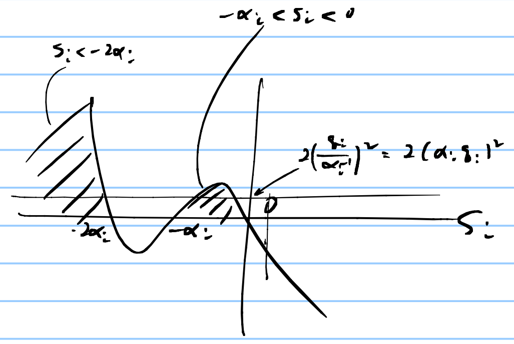
また $\lambda(\alpha_i)$ の傾きは

$$\lambda(\alpha_i)' = \frac{\alpha_i^{-1}-3}{2(\alpha_i+1)^2} = \frac{1-3\alpha_i}{2\alpha_i(\alpha_i+1)^2}$$

なので、$\alpha_i = \frac{1}{3}$ の右側で $\lambda(\alpha_i)' < 0$ なので、この停留点の値が全区間を通しての最大値となる。

$$\alpha_i^{-2}\left\{-s_i(s_i+\alpha_i)(s_i+2\alpha_i)+2\left(\frac{g_i}{\alpha_i^{-1}}\right)^2\right\}=0$$

$$s_i(s_i+\alpha_i)(s_i+2\alpha_i)=2\left(\frac{g_i}{\alpha_i^{-1}}\right)^2$$



$$\lambda(\alpha_i) = \frac{1}{2}\left[\ln\alpha_i - \ln(\alpha_i+s_i) + \frac{g_i^2}{\alpha_i+s_i}\right]$$

$$\frac{d\lambda}{d\alpha_i} = \frac{1}{2}\left[\frac{1}{\alpha_i} - \frac{1}{\alpha_i+s_i} + \frac{-g_i^2}{(\alpha_i+s_i)^2}\right]$$

$$= \frac{1}{2}\,\frac{(\alpha_i+s_i)^2 - \alpha_i(\alpha_i+s_i) - \alpha_i g_i^2}{\alpha_i(\alpha_i+s_i)^2}$$

$$= \frac{1}{2}\,\frac{\alpha_i^2 + 2\alpha_i s_i + s_i^2 - \alpha_i^2 - \alpha_i s_i - \alpha_i g_i^2}{\alpha_i(\alpha_i+s_i)^2}$$

$$= \frac{1}{2}\,\frac{\alpha_i(2s_i + \alpha_i^{-1}s_i^2 - s_i - g_i^2)}{\alpha_i(\alpha_i+s_i)^2}$$

$$= \frac{1}{2}\,\frac{\alpha^{-1}s_i^2 + s_i - g_i^2}{(\alpha_i+s_i)^2} = \frac{1}{2}\left\{\frac{\alpha^{-1}s_i^2}{(\alpha_i+s_i)^2} + \frac{s_i-g_i^2}{(\alpha_i+s_i)^2}\right\}$$

$$\frac{d^2\lambda}{d\alpha_i^2} = \frac{1}{2}\,\frac{s_i^2(-1)\alpha_i^{-2}(\alpha_i+s_i)^2 - \{\alpha_i^{-1}s_i^2 - (g_i^2-s_i)\}\,2(\alpha_i+s_i)}{(\alpha_i+s_i)^4}$$

$$= \frac{1}{2}\,\frac{-\alpha_i^{-2}s_i^2(\alpha_i+s_i)^2 - 2(\alpha_i+s_i)\{\alpha_i^{-1}s_i^2 - (g_i^2-s_i)\}}{(\alpha_i+s_i)^4}$$

$$= \frac{1}{2}\,\frac{-\alpha_i^{-2}s_i^2(\alpha_i+s_i) - 2\{\alpha_i^{-1}s_i^2 - (g_i^2-s_i)\}}{(\alpha_i+s_i)^3}$$

$$\frac{1}{2}\,\frac{-\alpha_i^{-2}(s_i^3 + 3\alpha_i s_i^2 + 2\alpha_i^2 s_i) + 2g_i^2}{(\alpha_i+s_i)^3} = \frac{1}{2}\,\frac{-s_i^3 - 3\alpha_i s_i^2 - 2\alpha_i^2 s_i + 2\alpha_i^2 g_i^2}{\alpha_i^3(\alpha_i+s_i)^3}$$

$$= \frac{1}{2}\,\frac{2(\alpha_i^2)(g_i^2-s_i) - 3\alpha_i s_i^2 - s_i^3}{(\alpha_i)^2(\alpha_i+s_i)^3}$$

$$= \frac{1}{2}\,\frac{\left(2(g_i^2-s_i)\alpha_i - s_i^2\right)\left(\alpha_i - \frac{1}{g_i^2-s_i}s_i^2\right) - \frac{s_i^4}{g_i^2-s_i} - s_i^3}{\alpha_i^2(\alpha_i+s_i)^3}$$

$$\alpha_i = \frac{3s_i^2 \pm \sqrt{9s_i^4 + 8(g_i^2-s_i)s_i^3}}{4(g_i^2-s_i)}$$

$$= \frac{3}{4}\,\frac{s_i^2}{g_i^2-s_i} \pm \frac{\quad}{4(g_i^2-s_i)}$$

$ax^2+bx+c=0$

$x = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$

$$\frac{s_i^2 - 2}{2(g_i^2 - s_i)}$$

$g_i^2 > s_i$

$$\frac{s_i^2}{2(g_i^2-s_i)} < \frac{1}{g_i^2-s_i}$$

$\alpha < 1$

⇓

$0 < 2 - s_i^2$

$\therefore s_i^2 < 2$

$\alpha > 1$

⇓

$0 > 2 - s_i^2$

$\therefore s_i^2 > 2$